12.1 Exponential Functions

Definition of the Exponential Function

The exponential function f with base b is defined by

$$f(x) = b^x \text{ or } y = b^x$$

where b is a positive constant other than 1 and x is any real number.

Example 1: The exponential function $f(x) = 13.49(0.967)^x - 1$ describes the number of 0-rings expected to fail, f(x), when the temperature is x°F. On the morning the Challenger was launched, the temperature was 31°F, colder than any previous experience. Find the number of 0-rings expected to fail at this temperature.

31°F = x°F

use x=31

$f(31) = 13.49(0.967)^{(31)} - 1$

$f(31) \approx 13.49(0.35336) - 1$

$f(31) \approx 4.7668 - 1$

$f(31) \approx 3.7668$

$f(31) \approx 4$

We would expect about 4 O-rings to fail.

Graphing Exponential Functions

Characteristics of Exponential Functions of the Form $f(x) = b^x$

1. The domain of $f(x) = b^x$ consists of all real numbers. The range consists of all positive real numbers.
2. The graphs of all exponential functions of the form $f(x) = b^x$ pass through the point (0,1) because $f(0) = b^0 = 1$. The y-intercept is (0,1).
3. The graph of $f(x) = b^x$ may be either strictly increasing or strictly decreasing:
   - If b > 1, $f(x) = b^x$ has a graph that goes up to the right and is an increasing function. The greater the value of b, the steeper the increase.
   - If b < 1, $f(x) = b^x$ has a graph that goes down to the right and is a decreasing function. The smaller the value of b, the steeper the decrease.
4. The graph of $f(x) = b^x$ approaches, but does not cross, the x-axis. The x-axis, or y = 0, is a horizontal asymptote.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

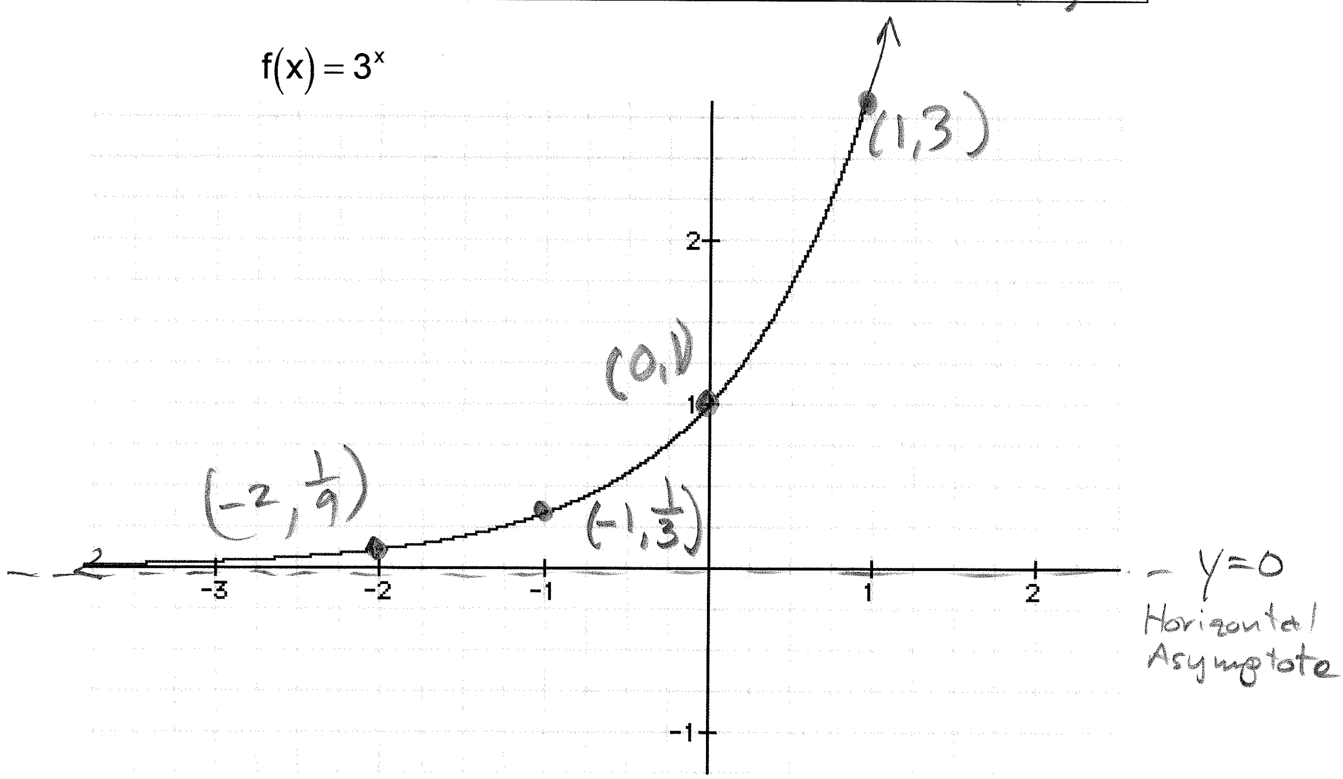
Example 2: Graph $f(x) = 3^x$

1. The domain is all real numbers. The range is all positive real numbers.
2. The y-intercept is (0,1).
3. Since b > 1, the graph is increasing.
4. y = 0 is a horizontal asymptote

Table of Coordinates

x	f(x)	(x, f(x))
−2	$f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
−1	$f(-1) = 3^{-1} = ?$ $\frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$f(0) = 3^0 = ?$ 1	(0, 1)
1	$f(1) = ?$ $3^1 = 3$	(1, 3)
2	$f(2) = 3^2 = 9$	(2, 9)

$f(x) = 3^x$



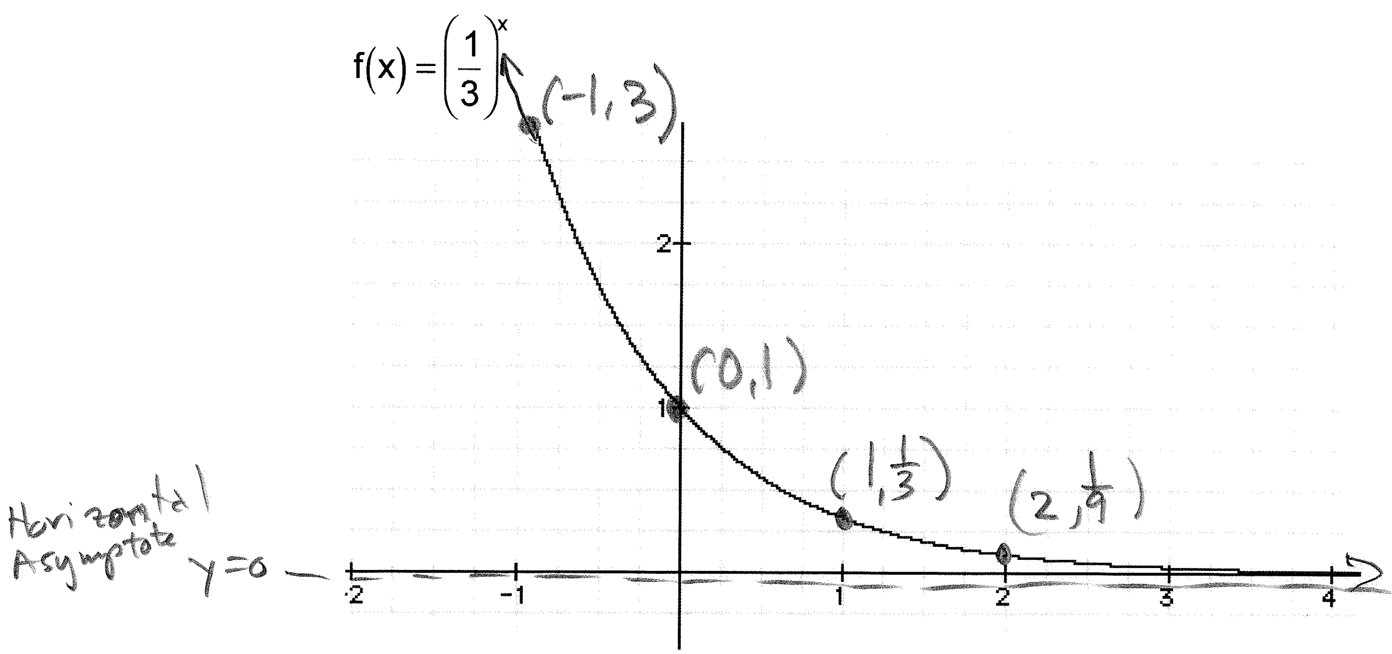
Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Example 3: Graph $f(x)=\left(\frac{1}{3}\right)^x$

1. The domain is all real numbers. The range is all positive real numbers.
2. The y-intercept is (0,1).
3. Since b < 1, the graph is decreasing.
4. y = 0 is a horizontal asymptote

Table of Coordinates

x	f(x)	(x, f(x))
–2	$f(-2)=\left(\frac{1}{3}\right)^{-2}=\frac{1}{3^{-2}}=9$	$(-2,9)$
–1	$f(-1)=\left(\frac{1}{3}\right)^{-1}=\frac{1}{3^{-1}}=?3$	$(-1,3)$
0	$f(0)=\left(\frac{1}{3}\right)^{0}=?1$	$(0,1)$
1	$f(1)=?\left(\frac{1}{3}\right)^1=\frac{1}{3}$	$(1,\frac{1}{3})$
2	$f(2)=\left(\frac{1}{3}\right)^2=\frac{1}{9}$	$(2,\frac{1}{9})$



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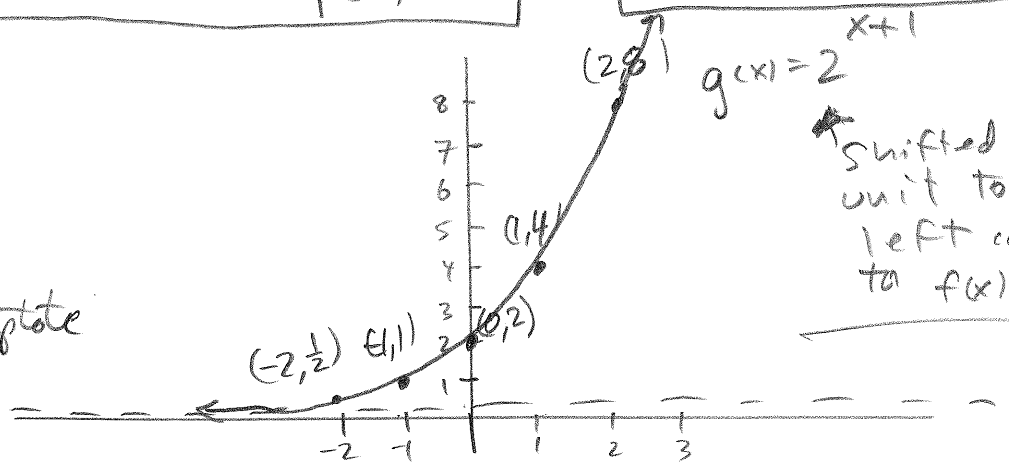
Example 4: Sketch the graph of $f(x)=2^x$ and $g(x)=2^{x+1}$ on the same coordinate axes. What is the relationship between the two graphs?

x	$f(x)=y$	$(x, f(x))$
-2	$f(-2)=2^{(-2)}=\frac{1}{2^2}=\frac{1}{4}$	$(-2,\frac{1}{4})$
-1	$f(-1)=2^{(-1)}=\frac{1}{2^1}=\frac{1}{2}$	$(-1,\frac{1}{2})$
0	$f(0)=2^{(0)}=1$	$(0,1)$
1	$f(1)=2^{(1)}=2$	$(1,2)$
2	$f(2)=2^{(2)}=4$	$(2,4)$

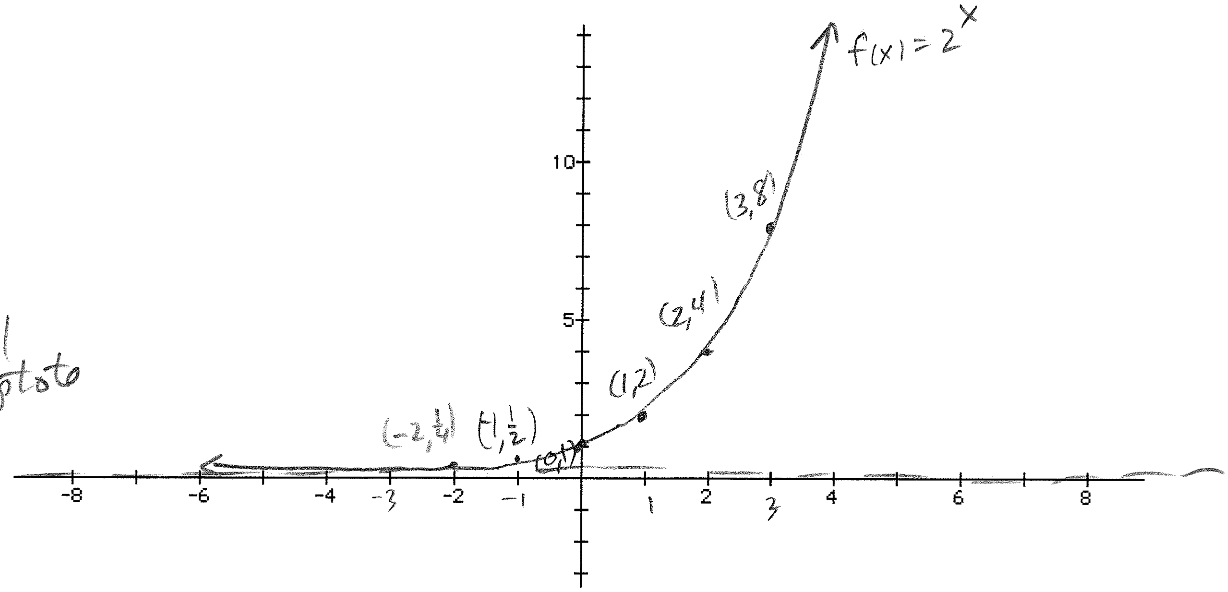
x	$g(x)=y$	$(x, g(x))$
-2	$g(-2)=2^{(-2)+1}=2^{-1}=\frac{1}{2}$	$(-2,\frac{1}{2})$
-1	$g(-1)=2^{(-1)+1}=2^0=1$	$(-1,1)$
0	$g(0)=2^{(0)+1}=2^1=2$	$(0,2)$
1	$g(1)=2^{(1)+1}=2^2=4$	$(1,4)$
2	$g(2)=2^{(2)+1}=2^3=8$	$(2,8)$



Shifted one unit to the left compared to $f(x)=2^x$.

Horizontal Asymptote

$y=0$



Horizontal Asymptote

$y=0$

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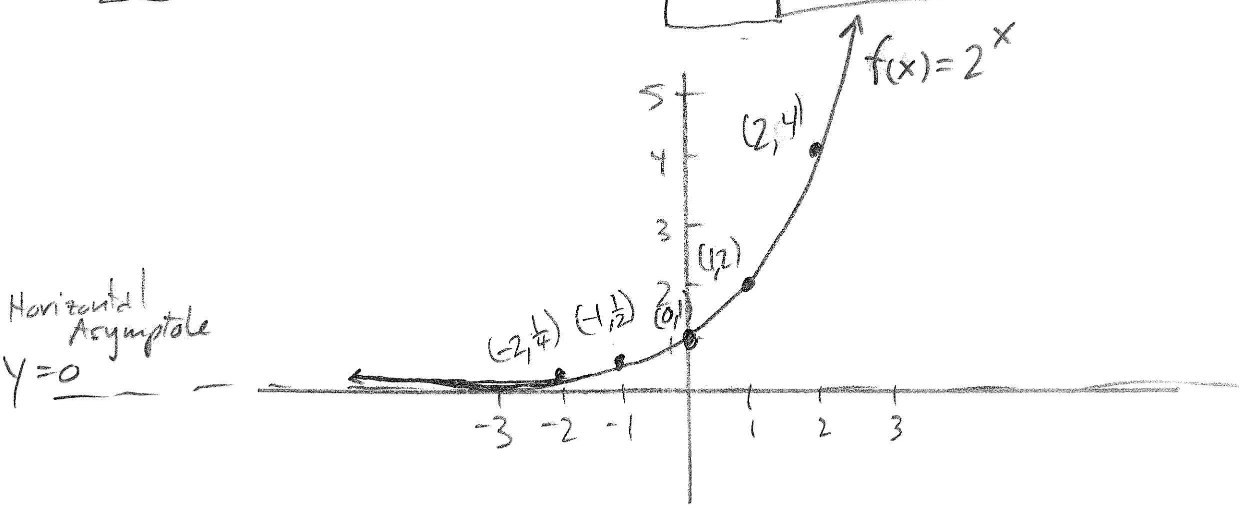
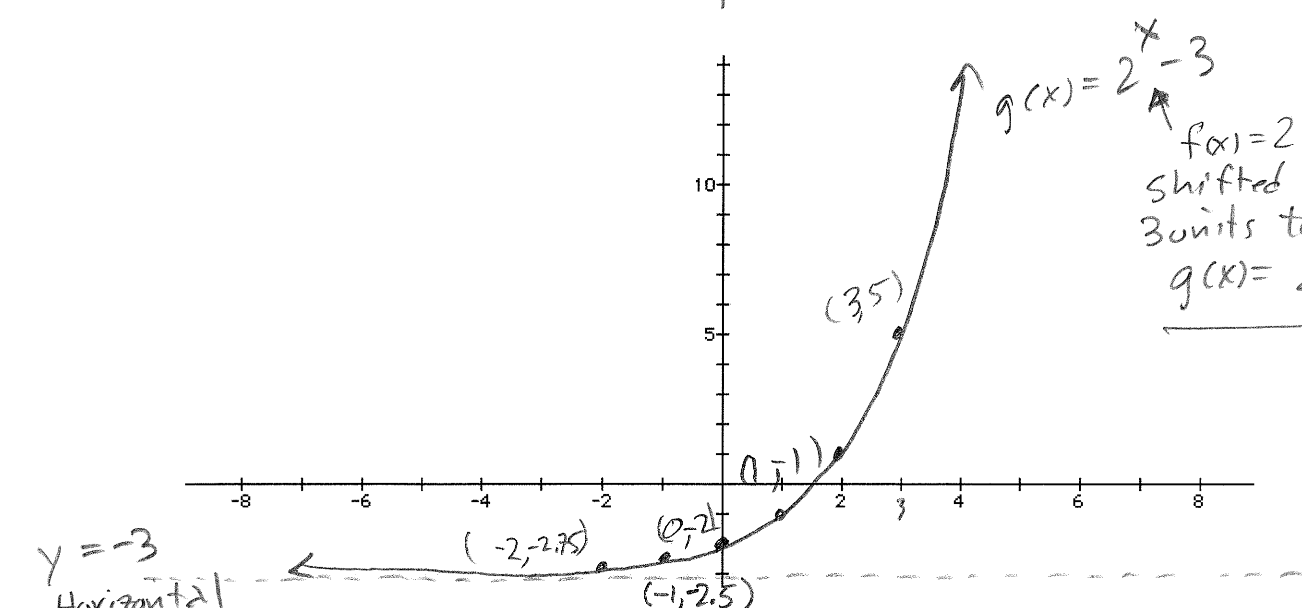
Example 5: Sketch the graph of $f(x) = 2^x$ and $g(x) = 2^x - 3$ on the same coordinate axes. What is the relationship between the two graphs?

X	f(x) = y	(x, f(x))
-2	$f(-2) = \frac{1}{4}$	$(-2, \frac{1}{4})$
-1	$f(-1) = \frac{1}{2}$	$(-1, \frac{1}{2})$
0	$f(0) = 1$	(0, 1)
1	$f(1) = 2$	(1, 2)
2	$f(2) = 4$	(2, 4)

X	g(x) = y	(x, g(x))
-2	$g(-2) = 2^{(-2)} - 3 = \frac{1}{4} - 3 = -2.75$	(-2, -2.75)
-1	$g(-1) = 2^{(-1)} - 3 = \frac{1}{2} - 3 = -2.5$	(-1, -2.5)
0	$g(0) = 2^{(0)} - 3 = 1 - 3 = -2$	(0, -2)
1	$g(1) = 2^{(1)} - 3 = 2 - 3 = -1$	(1, -1)
2	$g(2) = 2^{(2)} - 3 = 4 - 3 = 1$	(2, 1)





$f(x) = 2^x$ was shifted down 3 units to get $g(x) = 2^x - 3$.

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In general, the graph of $g(x) = b^x + d$ is simply the graph of $f(x) = b^x$ shifted d units up, if d > 0, or d units down, if d < 0. The horizontal asymptote of $g(x) = b^x + d$ is y = d.
The graph of $g(x) = b^{x+c}$ is simply the graph of $f(x) = b^x$ shifted c units left, if c > 0, or c units right, if c < 0. The horizontal asymptote of $g(x) = b^{x+c}$ is y = 0.

Example 6: For each of the following pairs of functions, tell how the graph of the second function can be obtained from the graph of the first function. Give the equation of the horizontal asymptote.

a. $f(x) = 4^x$, $g(x) = 4^{\overset{x+c?}{x-3}}$; $c = -3$, "Horizontal" Shift f(x) 3 units to the right to make $g(x) = 4^{x-3}$.

Horizontal Asymptote is $y = 0$

b. $f(x) = 4^x$, $g(x) = 4^{\overset{x+c?}{x+3}}$; $c = 3$, "Horizontal" shift f(x) 3 units to the left to make $g(x) = 4^{x+3}$

Horizontal Asymptote is $y = 0$

c. $f(x) = 4^x$, $g(x) = \overset{b^x + d?}{4^x + 2}$; $d = 2$, "Vertical" Shift f(x) 2 units upward to make $g(x) = 4^x + 2$

Horizontal Asymptote is $y = 2$

d. $f(x) = 4^x$, $g(x) = \overset{b^x + d?}{4^x - 3}$; $d = -3$, "Vertical" Shift f(x) 3 units downward to make $g(x) = 4^x - 3$

Horizontal Asymptote is $y = -3$

e. $f(x) = 4^x$, $g(x) = 4^{x+1} - 2$; $c = 1$ & $d = -2$
$= b^{x+c} + d$

Horizontal Asymptote is $y = -2$

"Horizontal" – Shift f(x) 1 unit to the left
"Vertical" – Shift f(x) 2 units downward
Combined, these two shifts will make $g(x) = 4^{x+1} - 2$

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$e \approx 2.71828182846$

The Natural Base e

An irrational number, symbolized by the letter e, appears as a base in many applied exponential functions. This irrational number is approximately equal to 2.72. The number e is called the natural base, and the function $f(x) = e^x$ is called the natural exponential function.

Example 7: The function $f(x) = 6e^{0.013x}$ describes world population, f(x), in billions, x years after 2000 subject to a growth rate of 1.3% annually. Use the function to find the world population in 2050.

x = years after 2000
x = 50 in 2,050

Find f(50):

$f(50) = 6e^{0.013(50)}$

$f(50) = 6e^{0.65}$

$f(50) \approx 6 \cdot (1.91554\ldots)$

$f(50) \approx 11.49324$

$f(50) \approx 11.5$

In 2,050, the world's population should be about 11.5 billion.

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Compound Interest

Formulas for Compound Interest
After t years, the balance, A, in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas:

1. For n compoundings per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

Example 8: Find the accumulated value of an investment of \$5000 for 10 years at an interest rate of 6.5% if the money is

$P = \$5,000$
$t = 10$ years
$r = 6.5\%$
$r = 0.065$

a. compounded semiannually ⟵ semiannually means $n=2$

$A = P\left(1+\frac{r}{n}\right)^{nt}$
$A = (\$5000)\left(1+\frac{0.065}{2}\right)^{[(2)\cdot(10)]}$
$A = \$5,000\,(1+0.0325)^{20}$
$A = \$5000\,(1.0325)^{20}$
$A \approx \$5,000\,(1.895838)$
$A \approx \$9,479.19$

b. compounded monthly ⟵ $n = 12$

$A = P\left(1+\frac{r}{n}\right)^{nt}$
$A = (\$5,000)\left(1+\frac{0.065}{12}\right)^{[(12)(10)]}$
$A \approx \$5,000\cdot(1+0.0054166666\ldots)^{120}$
$A \approx \$5000\cdot(1.00541667)^{120}$
$A \approx \$5,000\cdot(1.91218451\ldots)$
$A \approx \$5,000\cdot(1.9121845)$
$A \approx \$9,560.92$

c. compounded continuously.

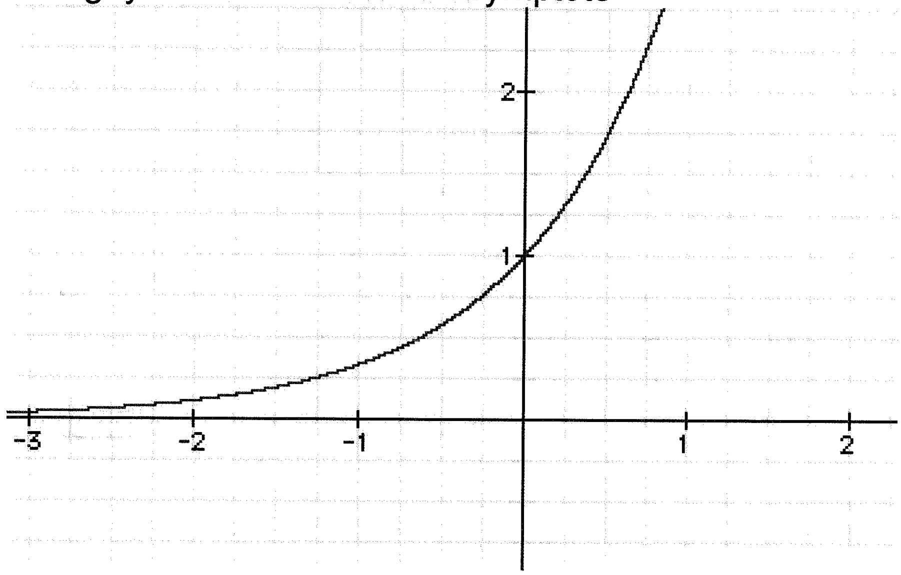
$A = Pe^{rt}$
$A = (\$5000)\,e^{(0.065)(10)}$
$A = \$5,000\,e^{0.65}$
$A \approx \$5,000\cdot(1.91554082901\ldots)$
$A \approx \$5,000\cdot(1.9155408)$
$A \approx \$9,577.704$
$A \approx \$9,577.70$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

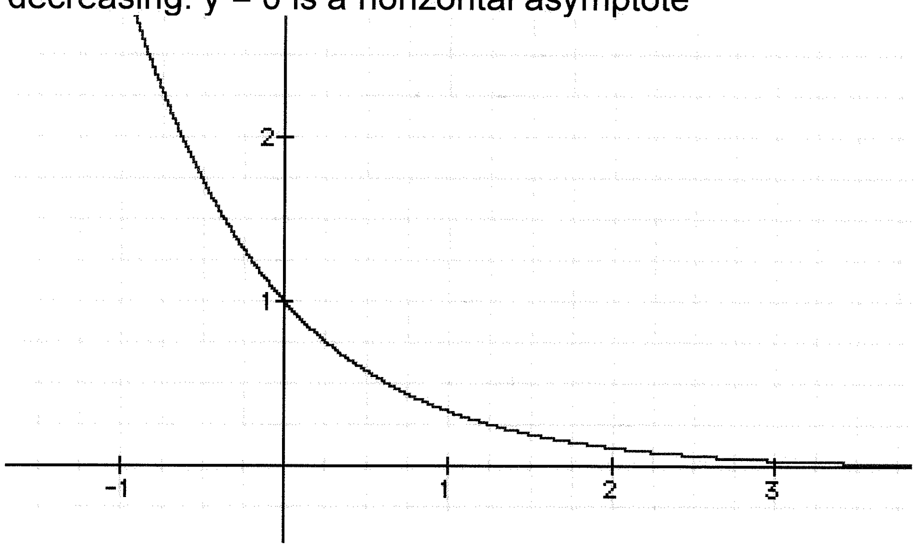
Answers Section 12.1

Example 1: f(31) ≅ 3.77. We would expect about 4 O-rings to fail.

Example 2: The domain is all real numbers. The range is all positive real numbers. The y-intercept is (0,1). Since b > 1, the graph is increasing. y = 0 is a horizontal asymptote



Example 3: The domain is all real numbers.The range is all positive real numbers.The y-intercept is (0,1). Since b < 1, the graph is decreasing. y = 0 is a horizontal asymptote



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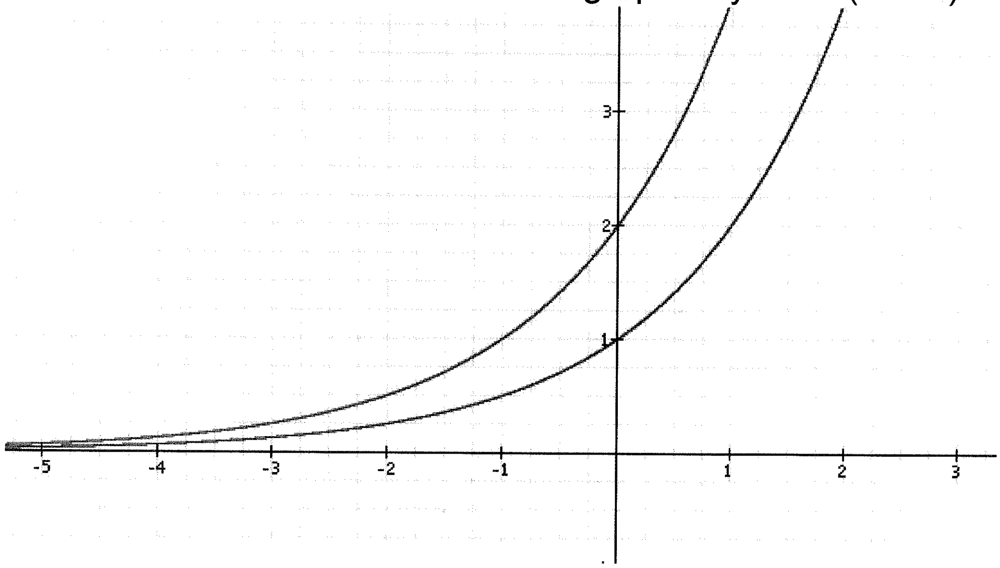
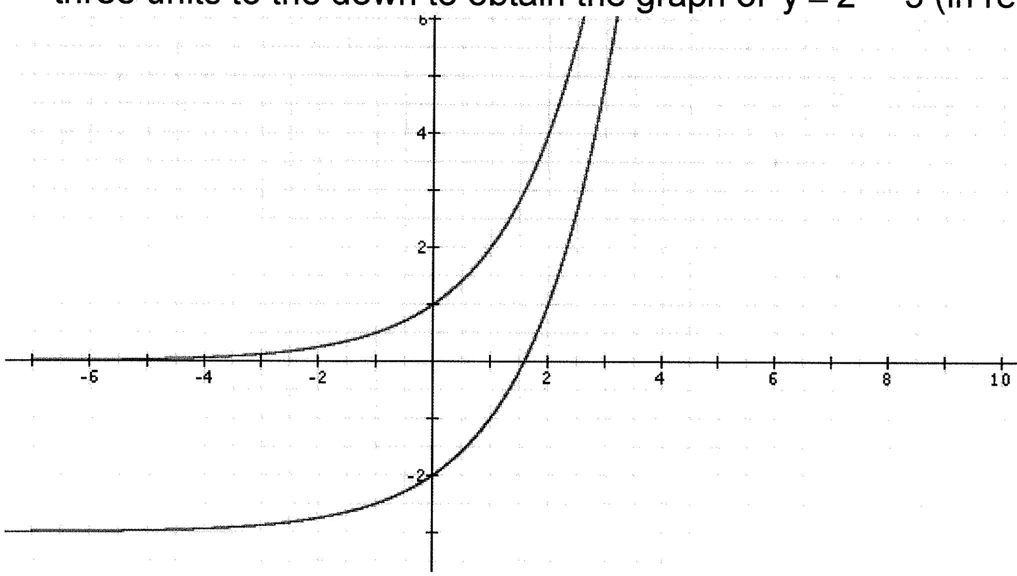
Example 4: Note that the graph of $y = 2^x$ (in purple) has been shifted one unit to the left to obtain the graph of $y = 2^{x+1}$ (in red)



Example 5: Note that the graph of $y = 2^x$ (in purple) has been shifted three units to the down to obtain the graph of $y = 2^x - 3$ (in red)



Example 6:

a. c = –3, shift 3 units right. Equation of horizontal asymptote: y = 0.
b. c = 3, shift 3 units left. Equation of horizontal asymptote: y = 0.
c. d = 2, shift 2 units up. Equation of horizontal asymptote: y = 2.
d. d = –3, shift 3 units down. Equation of horizontal asymptote: y = –3.

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e. $c = 1$ and $d = -2$, shift 1 unit left and 2 units down. Equation of horizontal asymptote: $y = -2$.

Example 7: $f(50) \cong 11.49$. The world population in 2050 is projected to be about 11.5 billion.

Example 8:

a. \$9,479.19
b. \$9,560.92
c. \$9,577.70

12.2 Logarithmic Functions

The Definition of Logarithmic Functions

For $x > 0$ and $b > 0$, $b \neq 1$

$y = \log_b x$ is equivalent to $b^y = x$.

The function $f(x) = \log_b x$ is the logarithmic function with base b.

Example 1: Write each equation in its equivalent exponential form.

a. $4 = \log_2 x$; $2^4 = x$

b. $-1 = \log_3 x$; $3^{-1} = x$

c. $\log_2 8 = y$; $2^y = 8$

Example 2: Write each equation in its equivalent logarithmic form.

a. $2^6 = x$; $\log_2(x) = 6$

b. $b^4 = 81$; $\log_b(81) = 4$

c. $2^y = 128$; $\log_2(128) = y$

Example 3: Evaluate each of the following.

a. $\log_{10} 100$; let $y = \log_{10}(100)$, $10^y = 100$

$10^y = 10^2$

$y = 2$, $\underline{\log_{10}(100) = 2}$

b. $\log_{25} 5$; let $y = \log_{25}(5)$, $25^y = 5$

$(5^2)^y = 5^1$

$5^{2y} = 5^1$

$2y = 1$

$\frac{2y}{2} = \frac{1}{2}$

$y = \frac{1}{2}$,

$\underline{\log_{25}(5) = \frac{1}{2}}$

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$c.\ \log_5 \frac{1}{5}$; let $y = \log_5\left(\frac{1}{5}\right)$, $5^y = \frac{1}{5}$

$5^y = 5^{-1}$

$y = -1$, $\underline{\log_5\left(\frac{1}{5}\right) = -1}$

$d.\ \log_2 \frac{1}{16}$; let $y = \log_2\left(\frac{1}{16}\right)$, $2^y = \frac{1}{16}$

$2^y = 16^{-1}$

$2^y = (2^4)^{-1}$

$2^y = 2^{-4}$

$y = -4$, $\underline{\log_2\left(\frac{1}{16}\right) = -4}$

SDWK

16

4 4

2 2 2 2

$16 = 2^4$

Basic Logarithmic Properties

Logarithmic Properties Involving One

1. $\log_b b = 1$ because 1 is the exponent to which b must be raised to obtain b. $(b^1 = b)$
2. $\log_b 1 = 0$ because 0 is the exponent to which b must be raised to obtain 1. $(b^0 = 1)$

Inverse Properties of Logarithms

For b > 0 and b ≠ 1,

$\log_b b^x = x$	The logarithm with base b of b raised to a power equals that power.
$b^{\log_b x} = x$	b raised to the logarithm with base b of a number equals that number.

Example 4: Evaluate each of the following.

a. $\log_8 8 = 1$

b. $\log_{1.5} 1.5 = 1$

c. $\log_8 1 = 0$

d. $\log_{1.7} 1 = 0$

e. $\log_8 8^5 = 5$

f. $\log_5 5^{2.3} = 2.3$

g. $7^{\log_7 8.3} = 8.3$

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x	$f(x)=3^x$	Point
-2	$f(-2)=3^{(-2)}=\frac{1}{3^2}=\frac{1}{9}$	$(-2,\frac{1}{9})$
-1	$f(-1)=3^{(-1)}=\frac{1}{3}$	$(-1,\frac{1}{3})$
0	$f(0)=3^{(0)}=1$	$(0,1)$
1	$f(1)=3^{(1)}=3$	$(1,3)$
2	$f(2)=3^{(2)}=9$	$(2,9)$

$g(x)=f^{-1}(x)$

TRADE x & y values

x	$g(x)=\log_3(x)=f^{-1}(x)$	Point
$\frac{1}{9}$	$g(\frac{1}{9})=\log_3(\frac{1}{9})=-2$	$(\frac{1}{9},-2)$
$\frac{1}{3}$	$g(\frac{1}{3})=\log_3(\frac{1}{3})=-1$	$(\frac{1}{3},-1)$
1	$g(1)=\log_3(1)=0$	$(1,0)$
3	$g(3)=\log_3(3)=1$	$(3,1)$
9	$g(3)=\log_3(9)=2$	$(9,2)$

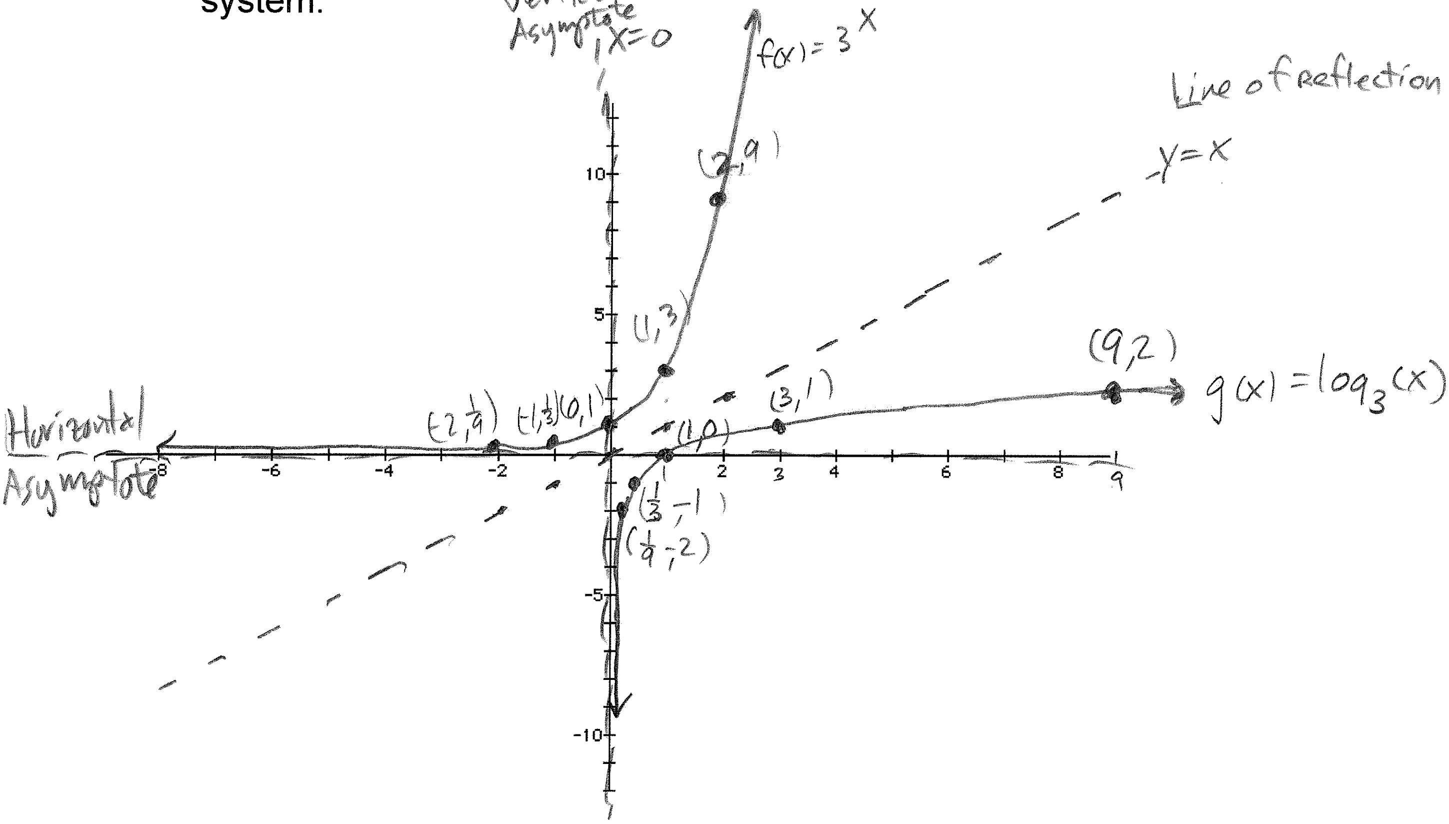
Graphs of Logarithmic Functions

The logarithmic function is the inverse of the exponential function with the same base. Thus the logarithmic function reverses the coordinates of the exponential function. The graph of the logarithmic function is the reflection of the exponential function about the line y=x.

Characteristics of the Graphs of Logarithmic Functions of the Form $f(x)=\log_b x$.

1. The x-intercept is 1. There is no y-intercept.
2. The y-axis is a vertical asymptote.
3. If b > 1, the function is increasing. If 0 < b < 1, the function is decreasing.
4. The graph is smooth and continuous. It has no sharp corners or gaps.

Example 5: Graph $f(x)=3^x$ and $g(x)=\log_3 x$ on the same coordinate system.



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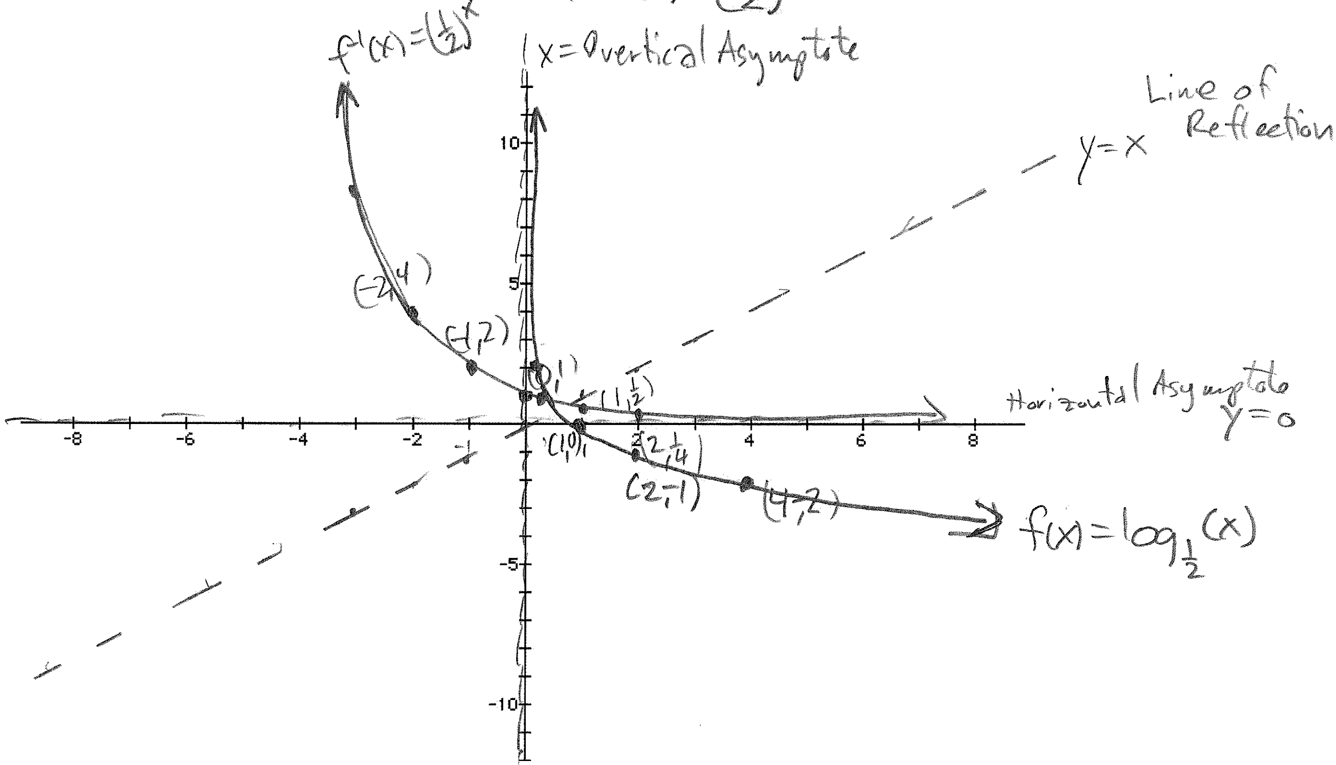
x	$f^{-1}(x)=(\frac{1}{2})^x$	Point
-2	$f^{-1}(-2)=(\frac{1}{2})^{(-2)}=(\frac{2}{1})^2=2^2=4$	(-2, 4)
-1	$f^{-1}(-1)=(\frac{1}{2})^{(-1)}=(\frac{2}{1})^1=2$	(-1, 2)
0	$f^{-1}(0)=(\frac{1}{2})^{(0)}=1$	(0, 1)
1	$f^{-1}(1)=(\frac{1}{2})^{(1)}=\frac{1}{2}$	(1, $\frac{1}{2}$)
2	$f^{-1}(2)=(\frac{1}{2})^{(2)}=\frac{1}{4}$	(2, $\frac{1}{4}$)

f(x) & $f^{-1}(x)$

TRADE x & y Values

x	$f(x)=\log_{\frac{1}{2}}(x)$	Point
4	$f(4)=\log_{\frac{1}{2}}(4)=-2$	(4, -2)
2	$f(2)=\log_{\frac{1}{2}}(2)=-1$	(2, -1)
1	$f(2)=\log_{\frac{1}{2}}(1)=0$	(1, 0)
$\frac{1}{2}$	$f(2)=\log_{\frac{1}{2}}(\frac{1}{2})=1$	($\frac{1}{2}$, 1)
$\frac{1}{4}$	$f(2)=\log_{\frac{1}{2}}(\frac{1}{4})=2$	($\frac{1}{4}$, 2)

Example 6: Graph $f(x)=\log_{\frac{1}{2}} x$ & $f^{-1}(x)=(\frac{1}{2})^x$



The Domain of a Logarithmic Function

In the expression $y=\log_b x$, x is the number produced when y is used as an exponent with base b, b > 0. Since b is always positive, x must also be positive. Thus the domain of the logarithmic function is x > 0, or all positive real numbers.

In general: domain of $f(x)=\log_b(x+c)$ consists of all x for which x + c > 0.

Example 7: Find the domain of the logarithmic function.

$f(x)=\log_5(x-7)$ Solve!

$x-7>0$

$x-7+7>7+0$

$x>7$

-7 0 7

Domain of $f(x)=\{x \mid x>7\}$

$=(7, \infty)$

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Common Logarithms

The logarithmic function with base 10 is called the common logarithmic function.

The function $f(x) = \log_{10} x$ is usually expressed simply as $f(x) = \log x$.

Most calculators have a "log" key that can be used to perform calculations with base-10 logarithms.
Logarithmic functions may be used to model some growth functions that start with rapid growth and then level off.

Example 8: The percentage of adult height attained by a girl who is x years old can be modeled by

$$f(x) = 62 + 35\log(x-4)$$

where x represents the girl's age (from 5 to 15) and f(x) represents the percentage of adult height. What percentage of adult height has a 10-year old girl attained?

For a 10-year old girl, $x = 10$.
Find $f(10)$:

$f(10) = 62 + 35\log[(10) - 4]$
$f(10) = 62 + 35\log(6)$
$f(10) \approx 62 + 35 \cdot [0.778\ldots]$
$f(10) \approx 62 + 27.23$
$f(10) \approx 89.23$

A 10-year old girl has attained about 89% of her adult height.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Natural Logarithms

The logarithmic function with base e is called the natural logarithmic function.

The function $f(x) = \log_e x$ is usually expressed simply as $f(x) = \ln x$.

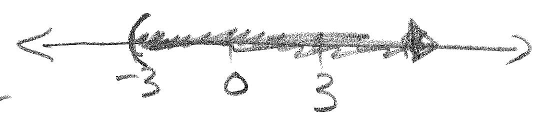
Most calculators have an "ln" key that can be used to perform calculations with base-e logarithms.

Example 9: Find the domain of the function.

a. $f(x) = \ln(x+3)$ Solve: $x+3>0$

$-3+x+3>-3+0$

$x>-3$



The Domain of $f(x) = \{x \mid x>-3\}$
$= (-3, \infty)$

Example 10: Simplify each expression.

a. $\ln e = 1$

b. $\ln e^4 = 4$

c. $e^{\ln 7} = 7$

d. $\ln e^{1.5x} = 1.5x$

e. $e^{\ln 3x} = 3x$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Summary of Properties of Logarithms

General Properties	Common Logarithms	Natural Logarithms
1. $\log_b 1 = 0$	$\log 1 = 0$	$\ln 1 = 0$
2. $\log_b b = 1$	$\log 10 = 1$	$\ln e = 1$
3. $\log_b b^x = x$	$\log 10^x = x$	$\ln e^x = x$
4. $b^{\log_b x} = x$	$10^{\log x} = x$	$e^{\ln x} = x$

You Must know these for our exam.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Answers Section 12.2

Example 1:

a. $x = 2^4$

b. $x = 3^{-1}$

c. $2^y = 8$

Example 2:

a. $\log_2 x = 6$

b. $\log_b 81 = 4$

c. $\log_2 128 = y$

Example 3:

a. $\log_{10} 100 = 2$

b. $\log_{25} 5 = \frac{1}{2}$

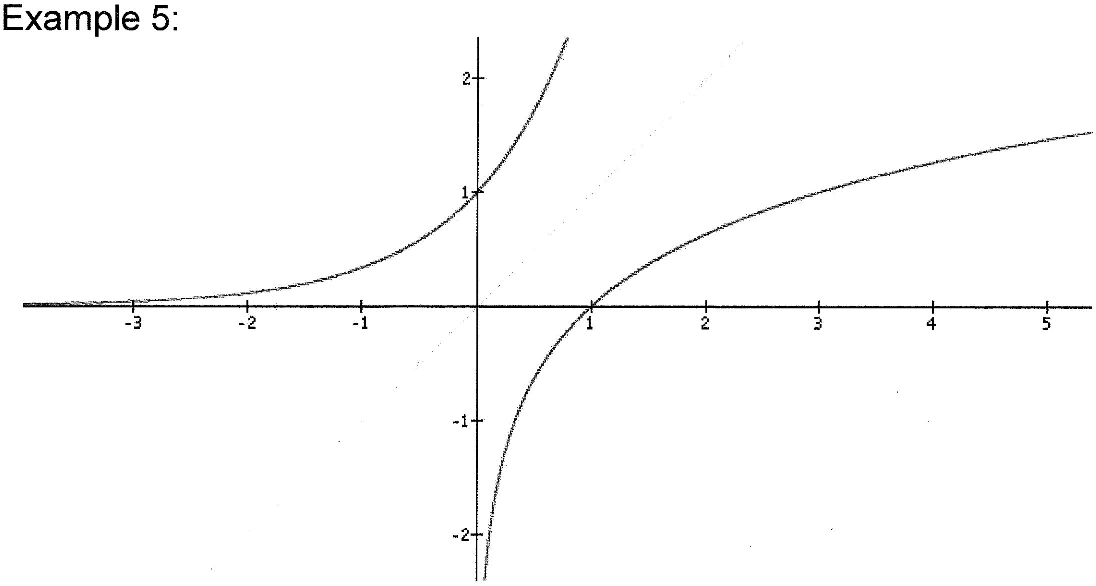
c. $\log_5 \frac{1}{5} = -1$

d. $\log_2 \frac{1}{16} = -4$

Example 4:

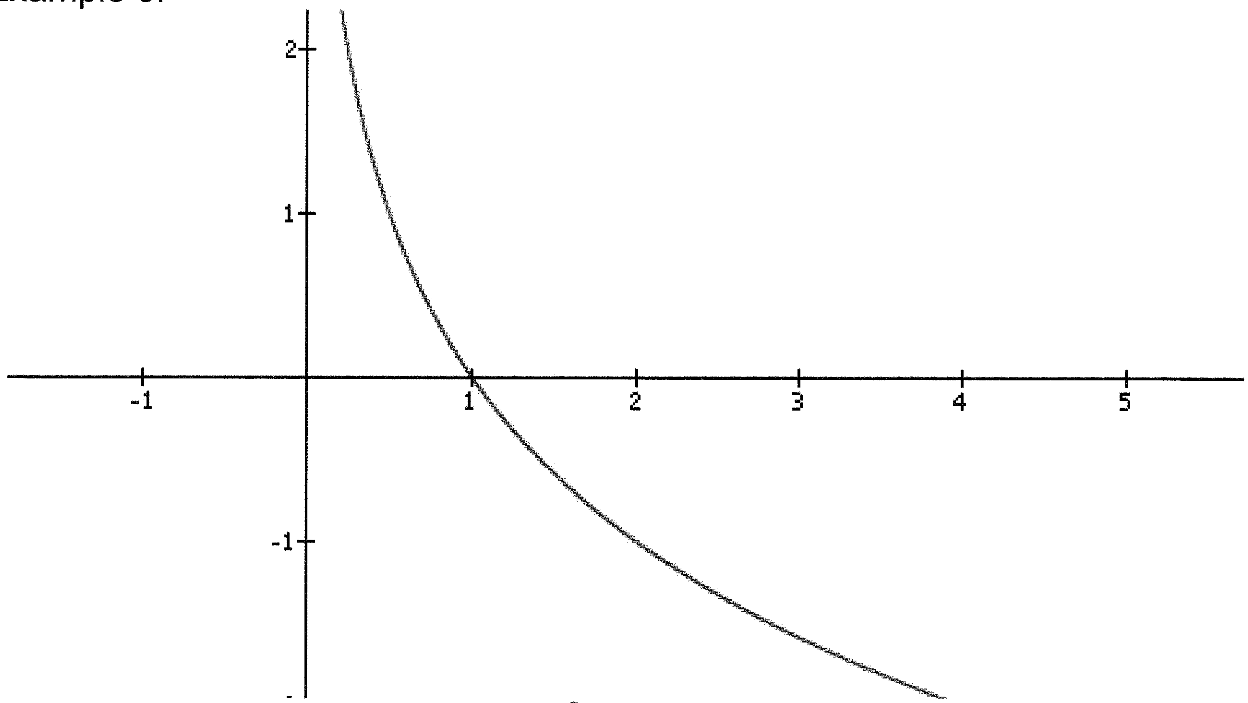
a. 1
b. 1
c. 0
d. 0
e. 5
f. 2.3
g. 8.3

Example 5:



Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Example 6:



Example 7: {x| x > 7}, or (7, ∞)

Example 8: f(10) ≅ 89.23 A 10-yr old girl has attained about 89% of adult height.

Example 9: {x| x > –3} , or (-3, ∞)

Example 10:
a. 1
b. 4
c. 7
d. 1.5x
e. 3x

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

12.3 Properties of Logarithms

The Product Rule

Let b, M and N be positive real numbers with $b \neq 1$.

$$\log_b MN = \log_b M + \log_b N$$

The logarithm of a product is the sum of the logarithms of the factors.

Example 1: Use the product rule to expand each logarithmic expression. Assume all variables and variable expressions represent positive numbers.

a. $\log_4(7x) = \log_4(7) + \log_4(x)$

b. $\log_4(7x(x-2)) = \log_4(7) + \log_4(x) + \log_4(x-2)$

c. $\log(10x) = \log(10) + \log(x) = 1 + \log(x)$

The Quotient Rule

Let b, M and N be positive real numbers with $b \neq 1$.

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

The logarithm of a quotient is the difference of the logarithms.

SDwk

8

2 4

2 2

$8 = 2^3$

Example 2: Use the quotient rule to expand each logarithmic expression. Assume all variables and variable expressions represent positive numbers.

a.. $\log_2 \frac{8}{x} = \log_2(8) - \log_2(x) = \log_2(2^3) - \log_2(x) = 3 - \log_2(x)$

b. $\log \frac{10^2}{5} = \log(10^2) - \log(5) = 2 - \log(5)$

c. $\ln \frac{8.7}{e^5} = \ln(8.7) - \ln(e^5) = \ln(8.7) - 5$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

The Power Rule

Let b and M be positive real numbers with $b \neq 1$, and let p be any real number.

$$\log_b M^p = p \log_b M$$

The logarithm of a number with an exponent is the product of the exponent and the logarithm of that number.

Example 3: Use the power rule to expand each logarithmic expression. Assume all variables and variable expressions represent positive numbers.

a. $\log_3 x^6 = 6 \cdot \log_3(x) = \underline{6\log_3(x)}$

b. $\log_2 (7x)^4 = 4 \cdot \log_2(7x) = \underline{4\log_2(7x)}$

c. $\log_6 \sqrt{x} = \log_6(x^{\frac{1}{2}}) = \frac{1}{2} \cdot \log_6(x) = \underline{\frac{1}{2}\log_6(x)}$

Expanding Logarithmic Expressions

Summary of Properties for Expanding Logarithmic Expressions

1. $\log_b MN = \log_b M + \log_b N$ Product Rule
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$ Quotient Rule
3. $\log_b M^p = p \log_b M$ Power Rule

Expanding logarithmic expressions may require that you use more than one property.

Example 4: Use logarithmic properties to expand each expression as much as possible. Assume all variables and variable expressions represent positive numbers.

a. $\log_2 (2x^2) = \log_2(2) + \log_2(x^2)$
$= 1 + 2 \cdot \log_2(x)$
$= \underline{1 + 2\log_2(x)}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

b. $\log \frac{10^{1.5}}{\sqrt{x}} = \log(10^{1.5}) - \log(\sqrt{x})$

$= (1.5)\cdot\log(10) - \log(x^{\frac{1}{2}})$

$= (1.5)\cdot 1 - \frac{1}{2}\cdot\log(x)$

$= \underline{1.5 - \frac{1}{2}\log(x)}$

c. $\log_b x^4\sqrt[3]{y} = \log_b(x^4) + \log_b(\sqrt[3]{y})$

$= 4\cdot\log_b(x) + \log_b(y^{\frac{1}{3}})$

$= 4\log_b(x) + \frac{1}{3}\cdot\log_b(y)$

$= \underline{4\log_b(x) + \frac{1}{3}\log_b(y)}$

d. $\log_4 \frac{\sqrt{x}}{25y^3} = \log_4(\sqrt{x}) - \log_4(25y^3)$

$= \log_4(x^{1/2}) - [\log_4(25) + \log_4(y^3)]$

$= \frac{1}{2}\cdot\log_4(x) - \log_4(25) - 3\cdot\log_4(y)$

$= \underline{\frac{1}{2}\log_4(x) - \log_4(25) - 3\log_4(y)}$

Condensing Logarithmic Expressions

To condense a logarithmic expression, we write a sum or difference of two logarithmic expressions as a single logarithmic expression. Use the properties of logarithms to do so.

Restatement of Properties of Logarithms:

1. $\log_b M + \log_b N = \log_b MN$	Product Rule
2. $\log_b M - \log_b N = \log_b \frac{M}{N}$	Quotient Rule
3. $p\log_b M = \log_b M^p$	Power Rule

Example 5: Write as a single logarithm. Assume all variables and variable expressions represent positive numbers.

a. $\log 25 + \log 4 = \log(25\cdot 4) = \log(100) = \log(10^2) = \underline{2}$

b. $\log 2x + \log 4 = \log(2x\cdot 4) = \underline{\log(8x)}$

c. $\log(x-1) + \log(x+4) = \underline{\log[(x-1)(x+4)]}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

d. $2\log x - \log 4 = \log(x^2) - \log(4) = \log\left(\frac{x^2}{4}\right)$

e. $7\log_4 5x - \log_4 8 = \log_4[(5x)^7] - \log_4(8) = \log_4\left[\frac{(5x)^7}{8}\right]$

f. $\frac{1}{2}\log_2 x + 2\log_2 5y^2 = \log_2(x^{1/2}) + \log_2[(5y^2)^2]$
$= \log_2(\sqrt{x}) + \log_2(5^2y^4)$
$= \log_2(\sqrt{x}) + \log_2(25y^4) = \log_2(25y^4\sqrt{x})$

The Change-of-Base Property

For any logarithmic bases a and b, and any positive number M,

$$\log_b M = \frac{\log_a M}{\log_a b}$$

The logarithm of M with base b is equal to the logarithm of M with any new base divided by the logarithm of b with that new base.
Since calculators generally have keys for only common or natural logs, the change of base formula must be used to evaluate logarithms with bases other than 10 or e.

If the new base, a, is chosen to be 10 or e, the change-of-base formula becomes:

$$\log_b M = \frac{\log M}{\log b} \qquad \text{or} \qquad \log_b M = \frac{\ln M}{\ln b}$$

Example 6: Evaluate each logarithm. Round your answer to the nearest hundredth.

a. $\log_2 133 = \frac{\log(133)}{\log(2)} \approx 7.05528\ldots \approx 7.06$

b. $\log_{0.5} 23.5 = \frac{\ln(23.5)}{\ln(0.5)} \approx -4.55458\ldots \approx -4.55$

c. $\log_6 458 = \frac{\ln(458)}{\ln(6)} \approx 3.41947\ldots \approx 3.42$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

$f(x) = \log_2(x-1)$

$f(x) = \dfrac{\ln(x-1)}{\ln(2)}$

Domain of $f(x) = \log(x^2)$

Solve! $x^2 > 0$

$x^2 = 0$

$x = 0$

True True

$x=-1$ 0 $x = 1$

Excluded

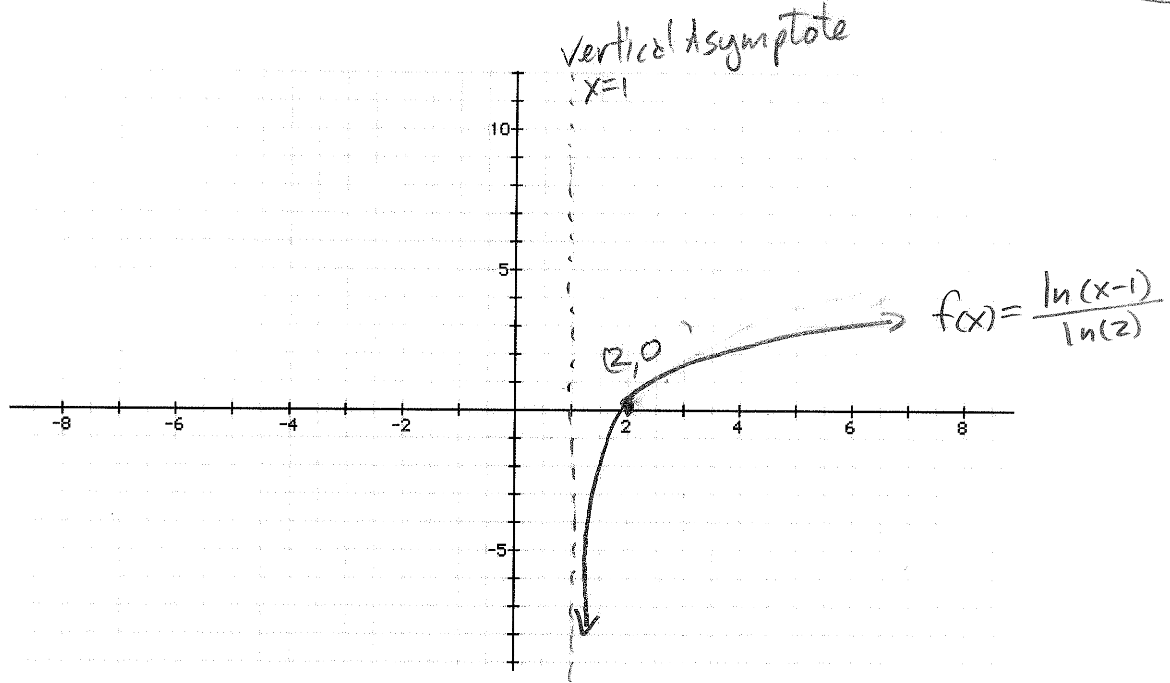
$(-1)^2 > 0$

$1 > 0$ TRUE

$(1)^2 > 0$

$1 > 0$ TRUE

Example 7: Use the change-of-base formula and your graphing calculator to graph $f(x) = \log_2(x-1)$. Indicate any vertical asymptotes with a dotted line.

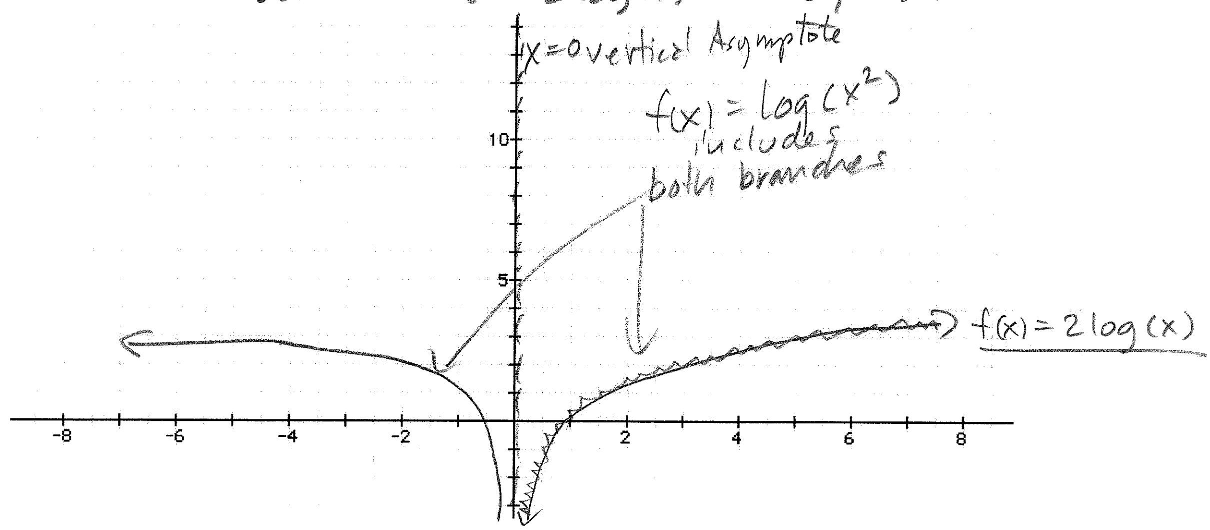


Example 8: Use your graphing calculator to graph $f(x) = 2\log x$ and $f(x) = \log x^2$. Show the graphs on the grid below Explain why the graphs are different. Domain of $f(x) = \log(x^2)$ is $(-\infty, 0) \cup (0, \infty)$. Domain of $f(x) = 2\log(x)$ is $(0, \infty)$.

Domain of $f(x) = 2\log(x)$

Solve: $x > 0$

0 1



Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Answers Section 12.3

Example 1:

a. $\log_4 7 + \log_4 x$

b. $\log_4 7 + \log_4 x + \log_4 (x-2)$

c. $1 + \log x$

Example 2:

a. $3 - \log_2 x$

b. $2 - \log 5$

c. $\ln 8.7 - 5$

Example 3:

a. $6 \log_3 x$

b. $4 \log_2 (7x)$

c. $\frac{1}{2} \log_6 x$

Example 4:

a. $1 + 2\log_2 x$

b. $1.5 - \frac{1}{2} \log x$

c. $4 \log_b x + \frac{1}{3} \log_b y$

d. $\frac{1}{2} \log_4 x - \log_4 25 - 3 \log_4 y$

Example 5:

a. 2

b. $\log(8x)$

c. $\log[(x-1)(x+4)]$

d. $\log\left(\frac{x^2}{4}\right)$

e. $\log_4\left(\frac{(5x)^7}{8}\right)$

f. $\log_2\left(25y^4\sqrt{x}\right)$

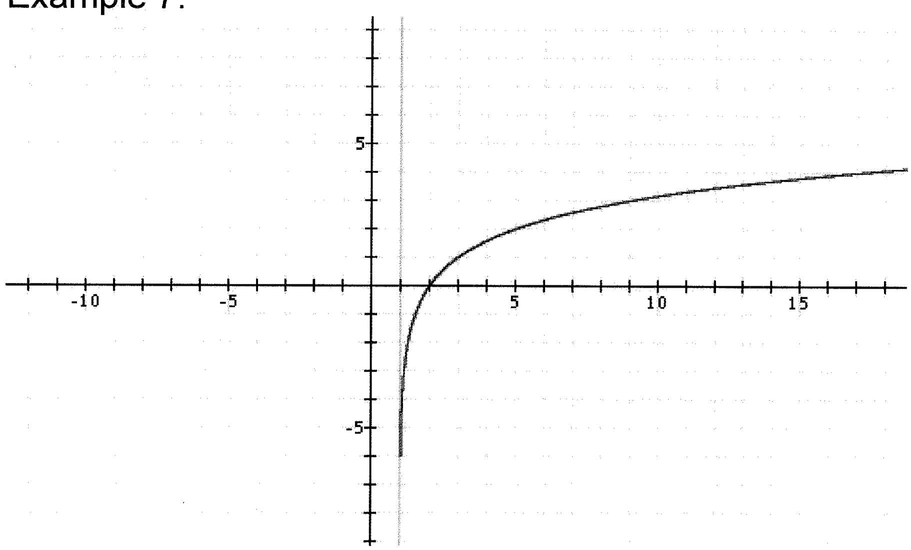
Example 6:

a. 7.06

b. –4.55

c. 3.42

Example 7:

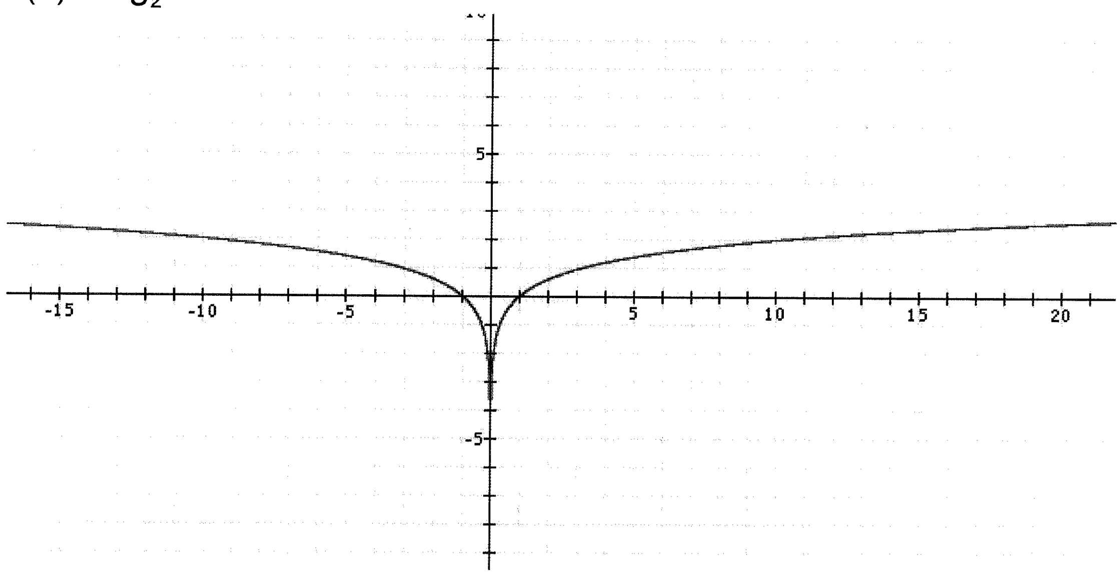


Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

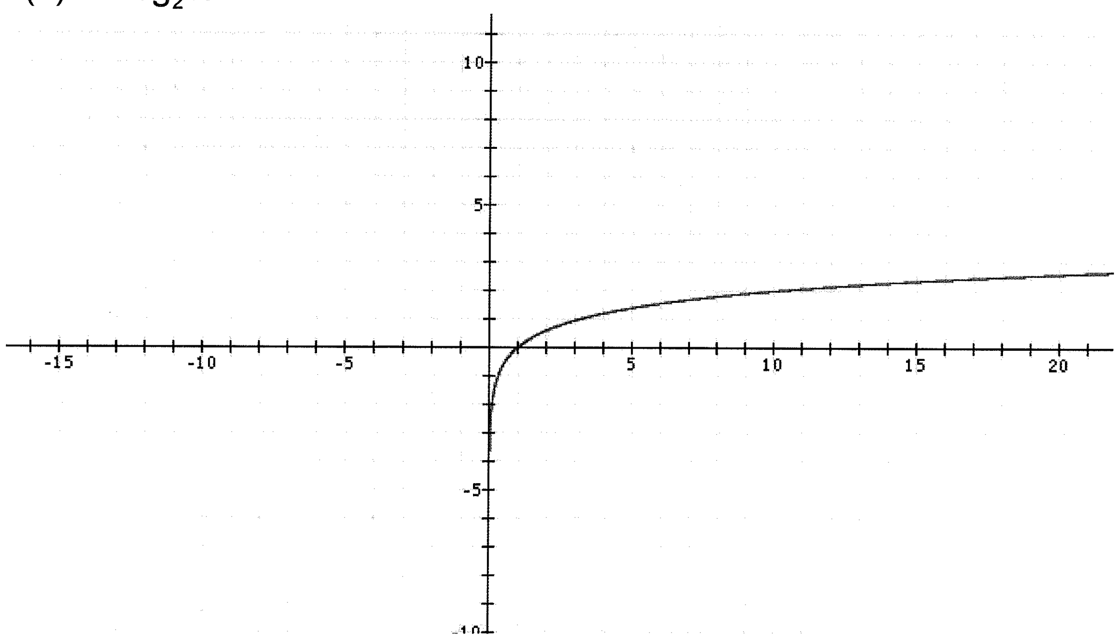
Example 8:

The domain of $f(x) = \log_2 x^2$ is the set of all real numbers except 0 but the domain of $f(x) = 2\log_2 x$ is the $\{x/\ x>0\}$.

$f(x) = \log_2 x^2$



$f(x) = 2\log_2 x$



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12.4 Exponential and Logarithmic Equations

Exponential Equations

An exponential equation is an equation containing a variable in an exponent. We solve exponential equations in by one of the following methods:

Method 1: Express both sides of the equation as a power of the same base.

1. Rewrite each side as a power of the same base.
2. Equate the exponents. (If $b^M = b^N$, then M=N. Note: b > 0.)
3. Solve the resulting equation.

Method 2: Take the natural logarithm of both sides of the equation.

1. Isolate the exponential expression.
2. Take the natural logarithm on both sides of the equation.
3. Simplify using one of the following properties:

   $\ln b^x = x\ln b$ or $\ln e^x = x$
4. Solve for the variable.

Example 1: Solve each exponential equation. Give exact answers.

SDWK

16

4 4

2 2 2 2

$16 = 2^4$

a. $2^{4x+1} = 16$

$2^{4x+1} = 2^4$

$4x+1=4$

$-1+4x+1=-1+4$

$4x=3$

$\frac{4x}{4}=\frac{3}{4}$

$x=\frac{3}{4}$

Check:

$2^{4(\frac{3}{4})+1}=16$

$2^{3+1}=16$

$2^4=16$

$16=16$

TRUE

The solution set is $\{\frac{3}{4}\}$.

SDWK

81

9 9

3 3 3 3

$81=3^4$

b. $3^{2x-1} = 81$

$3^{2x-1}=3^4$

$2x-1=4$

$1+2x-1=1+4$

$2x=5$

$\frac{2x}{2}=\frac{5}{2}$

$x=\frac{5}{2}$

check:

$3^{2(\frac{5}{2})-1}=81$

$3^{5-1}=81$

$3^4=81$

$81=81$

TRUE

The solution set is $\{\frac{5}{2}\}$.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

c. $4^{4x-1} = 8$

$(2^2)^{(4x-1)} = 2^3$

$2^{8x-2} = 2^3$

$8x - 2 = 3$

$2 + 8x - 2 = 2 + 3$

$8x = 5$

$\frac{8x}{8} = \frac{5}{8}$

$x = \frac{5}{8}$

check:

$4^{4(\frac{5}{8})-1} = 8$

$4^{\frac{5}{2}-1} = 8$

$4^{\frac{5}{2}-\frac{2}{2}} = 8$

$4^{\frac{3}{2}} = 8$

$(\sqrt[2]{4})^3 = 8$

$(2)^3 = 8$

$8 = 8$

TRUE!

The solution Set is $\{\frac{5}{8}\}$.

SDWK

4 → 2 2; 8 → 2 4 → 2 2

$\boxed{4 = 2^2}$

$\boxed{8 = 2^3}$

d. $9^{2x+3} = 27$

$(3^2)^{(2x+3)} = 3^3$

$3^{4x+6} = 3^3$

$4x + 6 = 3$

$-6 + 4x + 6 = -6 + 3$

$4x = -3$

$\frac{4x}{4} = \frac{-3}{4}$

$x = -\frac{3}{4}$

check:

$9^{2(\frac{-3}{4})+3} = 27$

$9^{\frac{-3}{2}+3} = 27$

$9^{\frac{-3}{2}+\frac{6}{2}} = 27$

$9^{\frac{3}{2}} = 27$

$(\sqrt[2]{9})^3 = 27$

$(3)^3 = 27$

$27 = 27$

TRUE

The solution set is $\{\frac{-3}{4}\}$.

SDWK

9 → 3 3; 27 → 3 9 → 3 3

$\boxed{9 = 3^2}$

$\boxed{27 = 3^3}$

Example 2: Solve each exponential equation. Give an exact answer, and then use your calculator to approximate your answer to two decimal places.

a. $10^x = 14$

$\ln(10^x) = \ln(14)$

$x \cdot \ln(10) = \ln(14)$

$\frac{x \cdot \ln(10)}{\ln(10)} = \frac{\ln(14)}{\ln(10)}$

$x = \frac{\ln(14)}{\ln(10)}$ ← EXACT Result. → $\left\{\frac{\ln(14)}{\ln(10)}\right\}$

$x \approx 1.1461280356\ldots$

$x \approx 1.15$

Check:

$10^{1.15} \approx 14$

$14.125.. \approx 14$

close ✓

The approximate solution set is $\{1.15\}$.

b. $4e^x = 17$

$\frac{4e^x}{4} = \frac{17}{4}$

$e^x = \frac{17}{4}$

$\ln(e^x) = \ln\left(\frac{17}{4}\right)$

$x = \ln\left(\frac{17}{4}\right)$

↑ Exact Result. → $\left\{\ln\left(\frac{17}{4}\right)\right\}$

$x \approx 1.44691898\ldots$

$x \approx 1.45$

Check:

$4e^{(1.45)} \approx 17$

$4 \cdot (4.26311..) \approx 17$

$17.705 \approx 17$

close ✓

The approximate solution set is $\{1.45\}$.

c. $15^{2+3x} = 122$

$\ln(15^{2+3x}) = \ln(122)$

$(2+3x) \cdot \ln(15) = \ln(122)$

$\frac{(2+3x) \cdot \ln(15)}{\ln(15)} = \frac{\ln(122)}{\ln(15)}$

$2 + 3x = \frac{\ln(122)}{\ln(15)}$

$-2 + 2 + 3x = \frac{\ln(122)}{\ln(15)} - 2$

$\frac{1}{3} \cdot \frac{3x}{1} = \left[\frac{\ln(122)}{\ln(15)} - 2\right] \cdot \frac{1}{3}$

$x = \frac{\ln(122)}{3\ln(15)} - \frac{2}{3}$ ← Exact Result. → $\left\{\frac{\ln(122)}{3\ln(15)} - \frac{2}{3}\right\}$

$x = \frac{1}{3} \cdot \left[\frac{\ln(122)}{\ln(15)} - 2\right]$

$x \approx \frac{1}{3} \cdot [1.77398 - 2]$

$x \approx \frac{1}{3} \cdot [-0.22602]$

$x \approx -0.07534$

$x \approx -0.08$

Check

$15^{2+3(-0.08)} \approx 122$

$15^{2-0.24} \approx 122$

$15^{1.76} \approx 122$

$117.468 \approx 122$

close ✓

The approximate solution set is $\{-0.08\}$.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

— SAME MEANING —

$\log_b(x) = y \Leftrightarrow b^y = x$

— Different Forms —

Logarithmic Equations

A logarithmic equation is an equation that contains a variable in a logarithmic expression. To solve a logarithmic equation:

1. Collect all of the terms involving logarithms on one side of the equation. Rewrite that logarithmic expression as a single logarithm using the properties of logarithms.
2. Rewrite the equation in its equivalent exponential form.
3. Solve the resulting equation.
4. Check proposed solutions and exclude any that produce the logarithm of a negative number or the logarithm of 0.

Example 3: Solve the given logarithmic equations. Give exact answers.

a. $\log_2 x = -4$

$2^{-4} = x$

$\frac{1}{2^4} = x$

$\frac{1}{16} = x$

check:

$\log_2\left(\frac{1}{16}\right) = -4$

$\log_2\left(\frac{1}{2^4}\right)$

$\log_2(2^{-4}) = -4$

$-4 = -4$

TRUE!

The solution set is $\left\{\frac{1}{16}\right\}$.

b. $\log_4(x+5) = 3$

$4^3 = x + 5$

$64 = x + 5$

$-5 + 64 = -5 + x + 5$

$59 = x$

check:

$\log_4[(59)+5] = 3$

$\log_4(64) = 3$

$\log_4(4^3) = 3$

$3 = 3$

TRUE!

The solution set is $\{59\}$.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

c. $\log_6(x+5)+\log_6 x=2$

$\log_6[(x+5)\cdot x]=2$

$\log_6(x^2+5x)=2$

The solution set is $\{4\}$.

$6^2=x^2+5x$

$36=x^2+5x$

$-36+36=-36+x^2+5x$

$0=x^2+5x-36$

$0=(x+9)(x-4)$

36
1, 36
2, 18
3, 12
4, 9
6, 6

Either

$x+9=0$, or $x-4=0$

$-9+x+9=-9+0$	$4+x-4=4+0$
$x=-9$	$x=4$

Check:

$\log_6[(-9)+5]+\log_6(-9)=2$

$\log_6(-4)+\log_6(-9)=2$

Not Allowed to use negative numbers in logs – Not in Domain

Check:

$\log_6[(4)+5]+\log_6(4)=2$

$\log_6(9)+\log_6(4)=2$

$\log_6(9\cdot 4)=2$

$\log_6(36)=2$

$\log_6(6^2)=2$

$2=2$

TRUE!

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Applications

Example 4: Use the formula $R = 6e^{12.77x}$, where x is the blood alcohol concentration and R, given as a percent, is the risk of having a car accident to find the blood alcohol concentration that corresponds to a 50% risk of having a car accident.

A 50% risk of having a car accident means that $R=50$, since R is given as a percent. Find x when $R=50$.

$$50 = 6e^{12.77x}$$

$$\frac{50}{6} = \frac{6e^{12.77x}}{6}$$

$$\frac{25 \cdot 2}{3 \cdot 2} = e^{12.77x}$$

$$\frac{25}{3} = e^{12.77x}$$

$$\ln\left(\frac{25}{3}\right) = \ln\left[e^{12.77x}\right]$$

$$\ln\left(\frac{25}{3}\right) = 12.77x$$

$$\frac{\ln\left(\frac{25}{3}\right)}{12.77} = \frac{12.77x}{12.77}$$

$$\frac{\ln\left(\frac{25}{3}\right)}{12.77} = x$$

$$\frac{2.12026}{12.77} \approx x$$

$$0.166 \approx x$$

$$0.17 \approx x$$

check

$$50 \approx 6e^{12.77\cdot(0.17)}$$

$$50 \approx 6e^{2.1709}$$

$$50 \approx 6\cdot(8.76617)$$

$$50 \approx 52.6$$

close ✓

A blood alcohol concentration of 0.17 corresponds to a 50% risk of having a car accident.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Example 5: If A is the accumulated value of an investment P after t years at r, the annual interest rate in decimal form and n, the number of compounding periods per year, then

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Find the accumulated value of an account in which $10,000 was invested for 10 years at 5% interest, compounded daily (360 times per year). $P = \$10{,}000$, $t = 10$ years, $r = 5\% = 0.05$, $n = 360$

$$A = (\$10{,}000)\left[1 + \frac{(0.05)}{(360)}\right]^{[(360)(10)]}$$

$$A \approx \$10{,}000\,[1 + 0.000138888889]^{3600}$$

$$A \approx \$10000\,[1.000138888889]^{3600}$$

$$A \approx \$10{,}000 \cdot (1.64866403039)$$

$$A \approx \$16{,}486.64$$

The accumulated value of this account should be $16,486.64.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Example 6: If A is the accumulated value of an investment P after t years at r, the annual interest rate in decimal form and with continuous compounding, then

$$A = Pe^{rt}$$

Find the accumulated value of $10,000 invested at 5% interest, compounded continuously for 10 years.

$P = \$10{,}000$, $r = 5\% = 0.05$, $t = 10$ years

$A = (\$10{,}000)\, e^{[(0.05)(10)]}$

$A = \$10{,}000\, e^{0.5}$

$A \approx \$10{,}000 \cdot (1.6487212707)$

$A \approx \$16{,}487.21$

The accumulated value of this account should be $16,487.21

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Answers Section 12.4

Example 1:

a. $\left\{\frac{3}{4}\right\}$

b. $\left\{\frac{5}{2}\right\}$

c. $\left\{\frac{5}{8}\right\}$

d. $\left\{-\frac{3}{4}\right\}$

Example 2:

a. {1.15}

b. {1.45}

c. {–0.08}

Example 3:

a. $\{\frac{1}{16}\}$

b. {59}

c. {4}

Example 4: 0.17, A blood alcohol content of .17 corresponds to a 50% risk of having a car accident.

Example 5: The accumulated value of an account in which $10,000 was invested for 10 years at 5% interest, compounded daily is $16,486.64.

Example 6 The accumulated value of $10,000 invested at 5% interest, compounded continuously for 10 years is $16,487.21.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

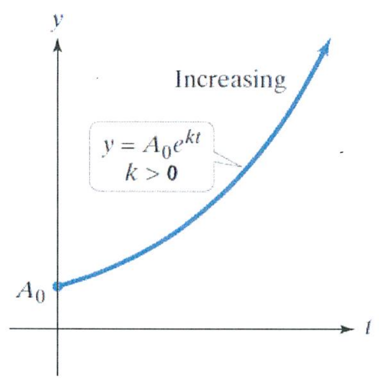
12.5 Exponential Growth and Decay; Modeling Data

Exponential Growth and Decay Models

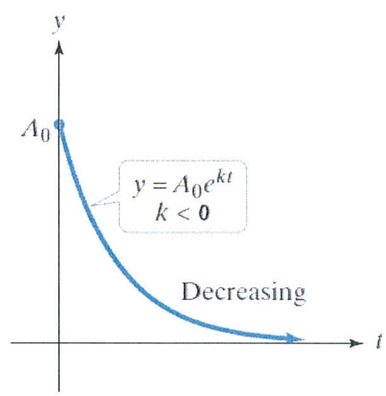
The mathematical model for **exponential growth** or **decay** is given by $f(t) = A_0 e^{kt}$, or $A = A_0 e^{kt}$.

• **If** $k > 0$ **the function models the amount, or size, of a** ***growing*** **entity.** A_0 is the original amount, or size, of the growing entity at time $t = 0$, A is the amount at time t and k is a constant representing the growth rate.

• **If** $k < 0$ **the function models the amount, or size, of a** ***decaying*** **entity.** A_0 is the original amount, or size, of the decaying entity at time $t = 0$, A is the amount at time t and k is a constant representing the decay rate.



(a) Exponential growth



(b) Exponential decay

Sometimes we need to use given data to determine k, the rate of growth or decay. After we compute the value of k, we can use the formula $A = A_0 e^{kt}$, to make predictions.

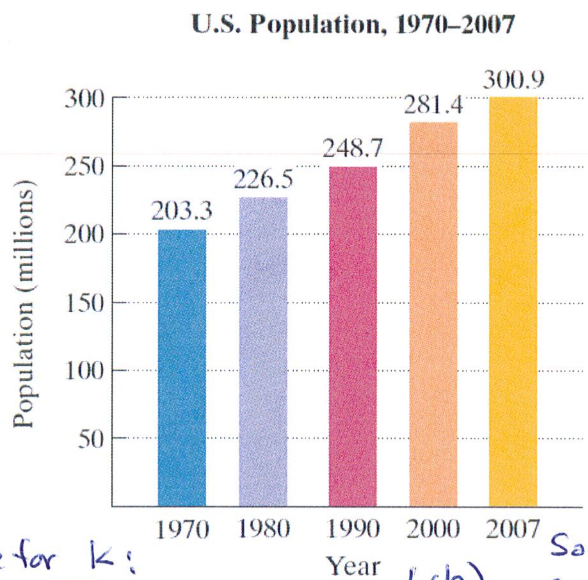
Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

$1.48 \approx e^{37k}$

$\log_e(1.48) \approx 37k$

☆ $\ln(1.48) \approx 37k$

Example 1: The graph below shows the U.S.population, in millions, for five selected years from 1970 through 2007. In 1970, the U.S. population was 203.3 million. By 2007, it had grown to 300.9 million.



a. Find an exponential growth function that models the data for 1970 through 2007.

b. By which year will the U.S. population reach 315 million?

a

Solve for k:

$A = A_0 e^{kt}$

$300.9 = 203.3e^{k(37)}$

$\frac{300.9}{203.3} = \frac{203.3e^{k(37)}}{203.3}$

☆ → $1.48 \approx e^{37k}$

$\ln(1.48) \approx \ln(e^{37k})$

$0.3921 \approx 37k$

$\frac{0.3921}{37} \approx \frac{37k}{37}$

$0.0106 \approx k$

$1.06\% \approx k$

$\boxed{f(t) = 203.3e^{0.0106t}}$

(b)

Solve for t!

$f(t) = 315$ million

$315 = 203.3e^{0.0106t}$

$\frac{315}{203.3} = \frac{203.3e^{0.0106t}}{203.3}$

$1.5494 \approx e^{0.0106t}$

$\ln(1.5494) \approx \ln(e^{0.0106t})$

$0.4379 \approx 0.0106t$

$\frac{0.4379}{0.0106} \approx \frac{0.0106t}{0.0106}$

$41.3 \approx t$

Since t=0 in 1970, we find that 41.3 corresponds to 2011. In 2011, we expect the U.S. population to be around 315 million.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Example 2: In 1990, the population of Africa was 643 million and by 2006 it had grown to 906 million. $t = 16$ in 2006

$A(16) = 906$ million

a. Use the exponential growth model $A = A_0 e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models the data. $t=0$ in 1990

$A_0 = 643$ million

b. By which year will Africa's population reach 2000 million, or two billion?

(a)

Solve for K →

$A = A_0 e^{kt}$

$906 = 643 e^{k(16)}$

$\frac{906}{643} = \frac{643 e^{16k}}{643}$

$1.409 \approx e^{16k}$

$\ln(1.409) \approx \ln(e^{16k})$

$\ln(1.409) \approx 16k$

$0.3429 \approx 16k$

$\frac{0.3429}{16} \approx \frac{16k}{16}$

$0.0214 \approx k$

$2.14\% \approx k$

↑ Growth constant

$A = A_0 e^{0.0214t}$

(b)

$2000 = 643 e^{0.0214t}$

$\frac{2000}{643} = \frac{643 e^{0.0214t}}{643}$

$3.1104 \approx e^{0.0214t}$

$\ln(3.1104) \approx \ln(e^{0.0214t})$

$\ln(3.1104) \approx 0.0214t$

$1.1348 \approx 0.0214t$

$\frac{1.1348}{0.0214} \approx \frac{0.0214t}{0.0214}$

$53.03 \approx t$

$\boxed{\text{In 2,043, Africa should have a population of 2 billion.}}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Our next example involves exponential decay and its use in determining the age of fossils and artifacts. The method is based on considering the percentage of carbon-14remaining in the fossil or artifact. Carbon-14 decays exponentially with a half-life of approximately 5715 years. The **half-life** of a substance is the time required for half of a given sample to disintegrate. Thus, after 5715 years a given amount of carbon-14 will have decayed to half the original amount. Carbon dating is useful for artifacts or fossils up to 80,000 years old. Older objects do not have enough carbon-14 left to determine age accurately.

Example 3: **a.** Use the fact that after 5715 years a given amount of carbon-14 will have decayed to half the original amount to find the exponential decay model for carbon-14. $A = \frac{A_0}{2}$

b. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found by an Arab Bedouin herdsman. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls. $A = 0.76A_0$

Solve for k:

(a)

$$A = A_0 e^{kt}$$

$$\frac{A_0}{2} = A_0 e^{k(5,715)}$$

$$\frac{1}{A_0} \cdot \frac{A_0}{2} = \frac{1}{A_0} \cdot \frac{A_0 e^{k(5,715)}}{1}$$

$$\frac{1}{2} = e^{5,715k}$$

$$\ln(\tfrac{1}{2}) = \ln(e^{5,715k})$$

$$\ln(\tfrac{1}{2}) = 5,715k$$

$$\frac{\ln(\frac{1}{2})}{5,715} = \frac{5,715k}{5,715}$$

$$\frac{\ln(\frac{1}{2})}{5,715} = k$$

$$\frac{-0.69315}{5,715} \approx k$$

$$-0.0001213 \approx k$$

$$-0.01213\% \approx k$$

$$\boxed{f(t) = A_0 e^{-0.0001213t}}$$

Solve for t:

$$0.76A_0 = A_0 e^{-0.0001213t}$$

$$\frac{0.76A_0}{A_0} = \frac{A_0 e^{-0.0001213t}}{A_0}$$

$$0.76 = e^{-0.0001213t}$$

$$\ln(0.76) = \ln(e^{-0.0001213t})$$

$$\ln(0.76) = -0.0001213t$$

$$\frac{\ln(0.76)}{-0.0001213} = \frac{-0.0001213t}{-0.0001213}$$

$$\frac{\ln(0.76)}{-0.0001213} = t$$

$$\frac{-0.27444}{-0.0001213} \approx t$$

$$2,262.46 \approx t$$

The Dead Sea Scrolls should about 2,262 years old.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Example 4: Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmospheric nuclear tests, we all have a measurable amount of strontium-90 in our bones.

$t = 28$, $A = \frac{1}{2}A_0$

a. Use the fact that after 28 years a given amount of strontium-90 will have decayed to half the original amount to find the exponential decay model for strontium-90. A_0 = original amount of strontium-90

b. Suppose that a nuclear accident occurs and releases 60 grams of strontium-90 into the atmosphere. How long will it take for strontium-90 to decay to a level of 10 grams.

a.

$$A = A_0 e^{kt}$$

Solve for k →

$$\frac{1}{2}A_0 = A_0 e^{k(28)}$$

$$\frac{\frac{1}{2}A_0}{A_0} = \frac{A_0 e^{28k}}{A_0}$$

$$\frac{1}{2} = e^{28k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{28k}\right)$$

$$\ln\left(\frac{1}{2}\right) = 28k$$

$$-0.6931 \approx 28k$$

$$\frac{-0.6931}{28} \approx \frac{28k}{28}$$

$$-0.0248 \approx k$$

$$-2.48\% \approx k$$

Decay Constant ↑

b.

$$A = A_0 e^{-0.0248t}$$

$$10 = 60 e^{-0.0248t}$$

← Solve for t:

$$\frac{10}{60} = \frac{60e^{-0.0248t}}{60}$$

$$\frac{1}{6} = e^{-0.0248t}$$

$$\ln\left(\frac{1}{6}\right) = \ln\left(e^{-0.0248t}\right)$$

$$\ln\left(\frac{1}{6}\right) = -0.0248t$$

$$-1.7918 \approx -0.0248t$$

$$\frac{-1.7918}{-0.0248} \approx \frac{-0.0248t}{-0.0248}$$

$$72.25 \approx t$$

After 72.25 years, 60 grams of Strontium-90 will decay to 10 grams.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

13.1 The Circle and Its Graph
MATH 64

Recall all Graphing we have covered:

a) Linear Equations $ax + by = c$

b) Quadratic Equations $ax^2 + bx + c = 0$

c) Exponential Equations $y = b^x$

d) Logarithmic Equations $y = \log_b(x)$

Pre-Requisite Knowledge:

1. You need to recall how to complete the square

$[\frac{1}{2}(6)]^2 = (3)^2 = 9$

$$x^2 + 6x$$
$$x^2 + 6x + 9 = (x+3)^2$$

2. You need to recall how to find the distance between two points in the coordinate plane.

> The **Distance Formula** is a formula used for computing the distance "d" between two points in a coordinate plane. If one point A is designated with coordinates (x_1, y_1) and the second point B is (x_2 , y_2), then
>
> **distance** "d" = AB = $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
>
> Always leave in simplified **radical** form – no decimals.

Ex.1 Find the distance between the point (–4, –3) and (2, 5).

(x_1, y_1) (x_2, y_2)

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$d = \sqrt{[(2)-(-4)]^2 + [(5)-(-3)]^2}$

$d = \sqrt{(2+4)^2 + (5+3)^2}$

$d = \sqrt{(6)^2 + (8)^2}$

$d = \sqrt{36 + 81}$

$d = \sqrt{117} = \sqrt{3^2} \cdot \sqrt{13}$

$d = 3\sqrt{13}$

SDWK

117

3 39

3 13

$117 = 3^2 \cdot 13$

Ex. 2 Find the distance between the point (3, –8) and (–4, 6).

(x_2, y_2) (x_1, y_1)

$d = \sqrt{[(-4)-(3)]^2 + [(-8)-(6)]^2}$

$d = \sqrt{(-7)^2 + (-14)^2}$

$d = \sqrt{49 + 196}$

$d = \sqrt{245}$

$d = \sqrt{7^2} \cdot \sqrt{5}$

$d = 7\sqrt{5}$

SDWK

245

5 49

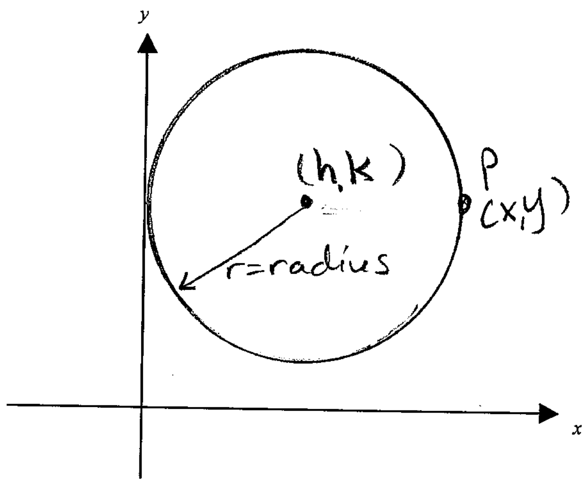
7 7

$245 = 7^2 \cdot 5$

Definition: A **circle** is the set of all points in a plane that are equidistant from a fixed point, called the center. The fixed distance from the circle's center to any point on the circle is called the radius. A compass is usually used to draw a circle (or an arc)

To find the Equation of a Circle

Step 1. Draw a circle in the rectangular coordinate system below.
Step 2. Label the center of the circle (h, k).
Step 3. Let (x, y) represent the coordinates of any point on the circle.



Step 4. What does the geometric definition (above) tell us about a point (x, y) on the circle?

The distance between the center (h,k) and P(x,y) is r.

Step 5. Use the distance formula to express the idea from step 4 algebraically.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$r = \sqrt{(x-h)^2 + (y-k)^2}$$
$$r^2 = [\sqrt{(x-h)^2 + (y-k)^2}]^2$$
$$r^2 = (x-h)^2 + (y-k)^2$$

$(x_1, y_1) = (h, k)$
$(x_2, y_2) = (x, y)$
$r = d$

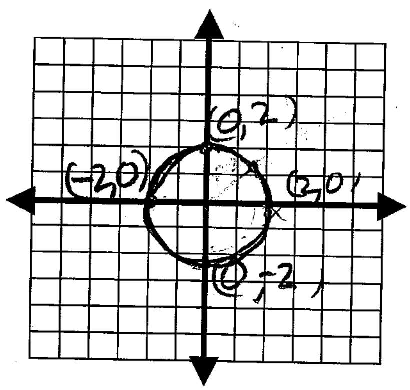
The Standard Form of the Equation of a Circle with center (h, k) and radius r.

$$(x-h)^2+(y-k)^2=r^2$$

① Write the standard form of the equation of a circle with center $(0, 0)$ and radius of 2. Graph the circle.

$(h,k)=(0,0)$
$r=2$

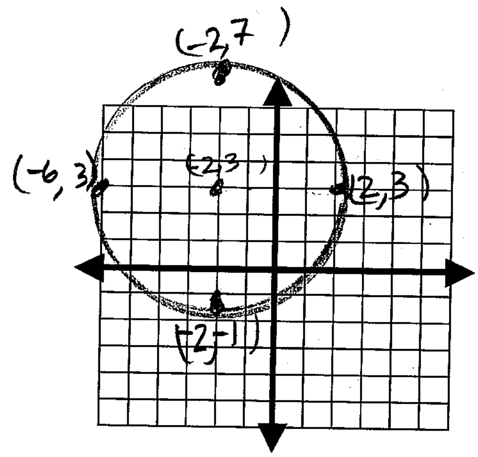
$(x-h)^2+(y-k)^2=r^2$
$[x-(0)]^2+[y-(0)]^2=(2)^2$
$x^2+y^2=4$



② Write the standard form of the equation of a circle with center $(-2, 3)$ and radius of 4. Graph the circle.

$(h,k)=(-2,3)$
$r=4$

$(x-h)^2+(y-k)^2=r^2$
$[x-(-2)]^2+[y-(3)]^2=(4)^2$
$(x+2)^2+(y-3)^2=16$

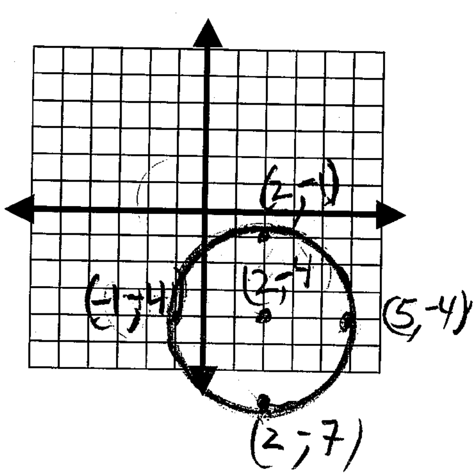


③ Find the center and radius of the circle whose equation is $(x-2)^2+(y+4)^2 = 9$. Graph the circle.

$h=2$, $k=-4$

Center $(h,k)=(2,-4)$

$r^2=9$
$r=3$

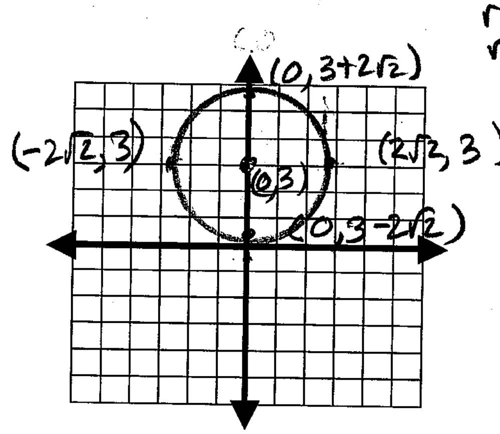


④ Find the center and radius of the circle whose equation is $x^2+(y-3)^2 = 8$. Graph the circle.

$h=0$, $k=3$

Center $(h,k)=(0,3)$

$r^2=8$
$r=\sqrt{8}$
$r=\sqrt{4}\cdot\sqrt{2}$
$r=2\sqrt{2}$



$(x-h)^2+(y-k)^2=r^2$

⑤ Write in standard form and graph: $x^2+y^2+4x-6y-23=0$.

$(x^2+4x)+(y^2-6y)-23+23=0+23$

$(x^2+4x+\underline{4})+(y^2-6y+\underline{9})=23+\underline{4}+\underline{9}$

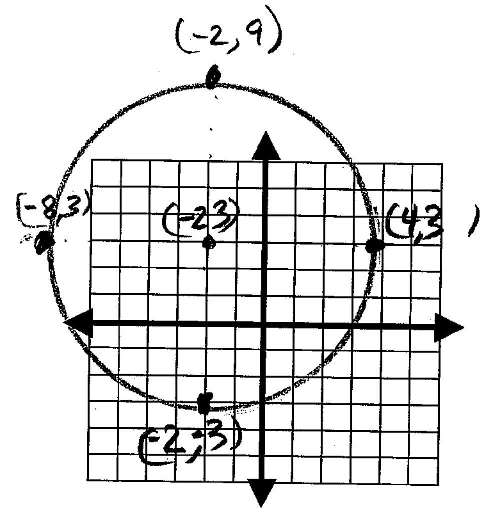
$[\frac{1}{2}(4)]^2=(2)^2=4$ $\quad [\frac{1}{2}(-6)]^2=(-3)^2=9$

$(x+2)^2+(y-3)^2=36$

$(h,k)=(-2,3)$

$r^2=36$

$r=6$



⑥ Write in standard form and graph: $x^2+y^2+12x+32=0$.

$(x^2+12x)+(y^2)+32-32=0-32$

$(x^2+12x+\underline{36})+y^2=-32+\underline{36}$

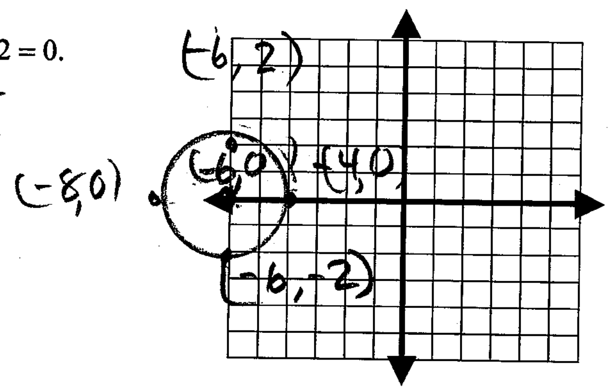
$[\frac{1}{2}(12)]^2=(6)^2=36$

$(x+6)^2+y^2=4$

$(h,k)=(-6,0)$

$r^2=4$

$r=2$

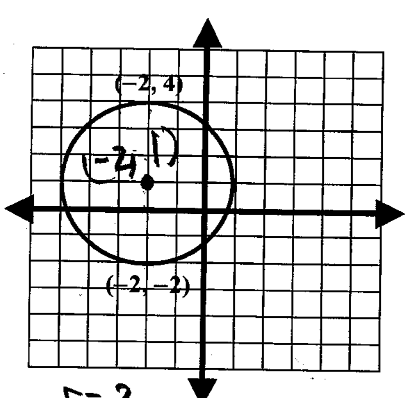


⑦ Find the equation of the circle graphed below.
Your answer should be in standard form.

$(x-h)^2+(y-k)^2=r^2$

$[x-(-2)]^2+[y-(1)]^2=(3)^2$

$(x+2)^2+(y-1)^2=9$



$r=3$

diameter $=6=2r$

radius $=r$

center $=(-2,1)$
$=(h,k)$

⑧ Graph the parabola: $y=x^2$ on the graph in problem ⑦. At what two points do the graphs intersect?

Systems of Non-Linear Equations Name ____________________
Date ______________

Recall SOME of the Equations we have covered:

a) Equations of Lines

$y = mx + b$

$ax + by = c$

b) Equations of Parabolas

$y = ax^2 + bx + c$

c) Equations of Circles

$(x-h)^2 + (y-k)^2 = r^2$

There are many other non-linear equations, such as an ellipse, hyperbola, sine, cosine, logistic, limacons, to name a few. For those of you continuing on in Mathematics there is so much to look forward to!

To be a SOLUTION TO A SYSTEM OF LINEAR EQUATIONS $\Leftrightarrow$ must work in BOTH !

Ex 1. Is $(-2, 3)$ a solution to the system ? Yes or No ?

$$\begin{cases} x + 2y = 4 \\ 2x + y = -1 \end{cases}$$

check: $(-2,3)$

$(-2)+2(3)=4$	$2(-2)+(3)=-1$
$-2+6=4$	$-4+3=-1$
$4=4$	$-1=-1$
TRUE!	TRUE!

Ex. 2. Is $(-1, 7)$ a solution to the system? Yes or No ?

$$\begin{cases} 3x + 2y = 11 \\ x + 5y = 36 \end{cases}$$

check: $(-1,7)$

$3(-1)+2(7)=11$	$(-1)+5(7)=36$
$-3+14=11$	$-1+35=36$
$11=11$,	$34=36$
TRUE!	False !

PREREQUISITE KNOWLEDGE:

Revisiting: Ex 1. Graph and find the solution to : $\begin{cases} x + 2y = 4 \\ 2x + y = -1 \end{cases}$

$x + 2y = 4$	$2x + y = -1$
$-x + x + 2y = -x + 4$	$-2x + 2x + y = -2x + (-1)$
$2y = -x + 4$	$y = -2x - 1$
$\frac{1}{2} \cdot \frac{2y}{1} = \frac{1}{2} \cdot (-x+4)$	$m = -2 = \frac{-2}{1}$
$y = -\frac{1}{2}x + 2$	$b = -1$

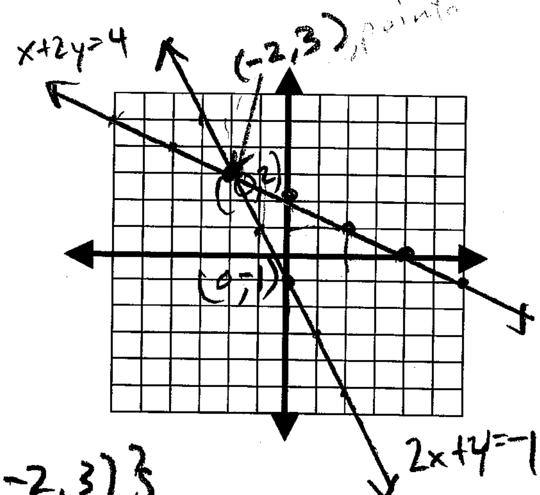
$m = -\frac{1}{2}$

$b = 2$

check: $(-2, 3)$

$(-2)+2(3)=4$	$2(-2)+(3)=-1$
$-2+6=4$	$-4+3=-1$
$4=4$	$-1=-1$
TRUE!	TRUE!

Point of Intersection is $(-2,3)$ point



The solution set is $\{(-2,3)\}$.

SOLVING A SYSTEM OF EQUATIONS

USING ELIMINATION AND SUBSTITUTION

SINCE GRAPHING A SYSTEM ONLY SHOWS LOCATION, CAN WE JUST SKIP THE GRAPHING AND USE ALGEBRA TO FIND THE POINT (if there is one) OF INTERSECTION?

☑ **YES**	THREE EASY STEPS	1. You want EACH equation to be in **standard** form. 2. You want to eliminate either the x **or** the y term or SUB 3. Solve and then find the point you need (substitute. . .AND CHECK)

Revisiting: Ex 1. $x + 2y = 4 \to -2\cdot(x+2y) = -2\cdot 4$; $-2x - 4y = -8$
$2x + y = -1$; $+\ 2x + y = -1$

$-3y = -9$

$\frac{-3y}{-3} = \frac{-9}{-3}$

$y = 3$

$-2\cdot(2x+y) = -2\cdot(-1)$

$-4x - 2y = 2$
$+\ \ x + 2y = 4$

$-3x = 6$

$\frac{-3x}{-3} = \frac{6}{-3}$

$x = -2$

check: $(-2,3)$

$(-2)+2(3)=4$
$-2+6=4$
$4=4$, TRUE!

$2(-2)+(3)=-1$
$-4+3=-1$
$-1=-1$
TRUE

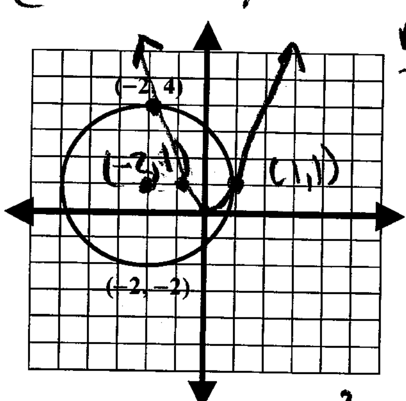
The solution set is $\{(-2,3)\}$.

A SOLUTION TO A SYSTEM OF NON-LINEAR EQUATIONS ⇔ must work in BOTH !

Recall this problem from our last set of notes

⑦ Find the equation of the circle graphed below.
Your answer should be in standard form.

center $= (-2,1) = (h,k)$
$(x-h)^2 + (y-k)^2 = r^2$
$r = 3$



$[x-(-2)]^2 + (y-1)^2 = (3)^2$
$(x+2)^2 + (y-1)^2 = 9$

⑧ Graph the parabola: $y = x^2$ on the graph in problem ⑦. At what two points do the graphs intersect?

$(-2,4)$ & $(1,1)$

check: $(-2,4)$

$y = x^2$ & $(x+2)^2 + (y-1)^2 = 9$

$(4) = (-2)^2$
$4 = 4$
TRUE!

$[(-2)+2]^2 + [(4)-1]^2 = 9$
$(0)^2 + (3)^2 = 9$
$9 = 9$
TRUE!

check: $(1,1)$

$(1) = (1)^2$
$1 = 1$
TRUE!

$[(1)+2]^2 + [(1)-1]^2 = 9$
$(3)^2 + 0^2 = 9$
$9 = 9$
TRUE!

The solution ~~set~~ is $\{(-2,4),(1,1)\}$.

13.5 Systems of Non-Linear Equations

A system of two non-linear equations in two variables, also called a nonlinear system, contains at least one equation that cannot be expressed as $Ax + By = C$. We solve systems by using either elimination or substitution.

Ex. 1 $\begin{cases} x^2 = 2y + 10 \\ 3x - y = 9 \end{cases}$

The solution set is $\{(2,-3), (4,3)\}$.

Use $3x - y = 9$ for substitution.

$y + 3x - y = y + 9$

$-9 + 3x = y + 9 - 9$

$\underline{3x - 9 = y}$

$x^2 = 2y + 10$

$x^2 = 2(3x-9) + 10$

$x^2 = 6x - 18 + 10$

$(-6x+8) + x^2 = 6x - 8 + (-6x+8)$

$x^2 - 6x + 8 = 0$ $\boxed{\frac{8}{1,8 \;\; 2,4}}$

$(x-2)(x-4) = 0$

Either $x - 2 = 0$, or $x - 4 = 0$

$\underline{x = 2}$, or $\underline{x = 4}$

Find y!

$3x - 9 = y$	$3x - 9 = y$
$\underline{x = 2}$	$\underline{x = 4}$
$3(2) - 9 = y$	$3(4) - 9 = y$
$6 - 9 = y$	$12 - 9 = y$
$-3 = y$	$3 = y$
$(2,-3)$	$(4,3)$

check: $(2,-3)$

$(2)^2 = 2(-3) + 10$

$4 = -6 + 10$

$4 = 4$ TRUE!

$3(2) - (-3) = 9$

$6 + 3 = 9$

$9 = 9$, TRUE!

Check: $(4,3)$

$(4)^2 = 2(3) + 10$

$16 = 6 + 10$

$16 = 16$ TRUE!

$3(4) - (3) = 9$

$12 - 3 = 9$

$9 = 9$, TRUE!

Ex. 2 $\begin{cases} (x-2)^2 + (y+3)^2 = 4 \\ x - y = 3 \end{cases}$

The solution set is $\{(0,-3), (2,-1)\}$

use $x - y = 3$ for substitution.

$x - y + y = y + 3$

$\underline{x = y + 3}$

$(x-2)^2 + (y+3)^2 = 4$

$[(y+3)-2]^2 + (y+3)^2 = 4$

$(y+1)^2 + (y+3)^2 = 4$

$(y+1)(y+1) + (y+3)(y+3) = 4$

$y^2 + 2y + 1 + y^2 + 6y + 9 = 4$

$2y^2 + 8y + 10 = 4$

$2y^2 + 8y + 10 - 4 = 4 - 4$

$2y^2 + 8y + 6 = 0$

$\frac{1}{2}(2y^2 + 8y + 6) = \frac{1}{2} \cdot 0$

$y^2 + 4y + 3 = 0$ $\boxed{\frac{3}{1,3}}$

$(y+3)(y+1) = 0$

Either

$y + 3 = 0$, or $y + 1 = 0$

$\underline{y = -3}$, or $\underline{y = -1}$

Find x!

$x = y + 3$	$x = y + 3$
$\underline{y = -3}$	$\underline{y = -1}$
$x = (-3) + 3$	$x = (-1) + 3$
$x = 0$	$x = 2$
$(0,-3)$	$(2,-1)$

check: $(0,-3)$

$[(0)-2]^2 + [(-3)+3]^2 = 4$

$(-2)^2 + (0)^2 = 4$

$4 + 0 = 4$

$4 = 4$, TRUE!

$(0) - (-3) = 3$

$3 = 3$, TRUE!

check: $(2,-1)$

$[(2)-2]^2 + [(-1)+3]^2 = 4$

$(0)^2 + (2)^2 = 4$

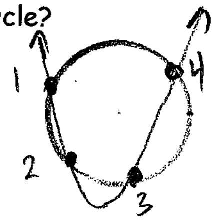
$4 = 4$, TRUE!

$(2) - (-1) = 3$

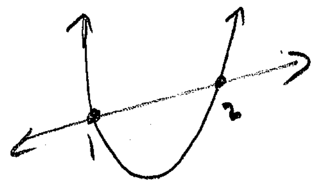
$2 + 1 = 3$

$3 = 3$, TRUE!

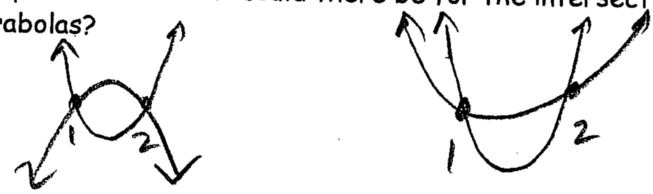
Ex 3. How many possible solutions could there be for the intersection of a parabola and a circle?



Ex 4. How many possible solutions could there be for the intersection of a parabola and a line?



Ex 5. How many possible solutions could there be for the intersection of two parabolas?



Ex 6. Solve the following system of equations: $\begin{cases} y=-x^2-2x+14 \\ y=x^2-4x-10 \end{cases}$

use $y=-x^2-2x+14$ for substitution.

$y=x^2-4x-10$

$-x^2-2x+14=x^2-4x-10$

$-x^2-2x+14+x^2+2x-14=x^2-4x-10+x^2+2x-14$

$0=2x^2-2x-24$

$\frac{1}{2}\cdot 0=\frac{1}{2}\cdot(2x^2-2x-24)$

$0=x^2-x-12$

$0=(x-4)(x+3)$

12
1,12
2,6
3,4

Either

$x-4=0$, or $x+3=0$

$x=4$, or $x=-3$

Find y! $x=4$

$y=-x^2-2x+14$

$x=4$

$y=-(4)^2-2(4)+14$

$y=-16-8+14$

$y=-24+14$

$y=-10$

$(4,-10)$

$x=-3$

$y=-(-3)^2-2(-3)+14$

$y=-9+6+14$

$y=-3+14$

$y=11$

$(-3,11)$

check! $(4,-10)$

$y=-x^2-2x+14$

$-10=-(4)^2-2(4)+14$

$-10=-16-8+14$

$-10=-24+14$

$-10=-10$, TRUE!

$y=x^2-4x-10$

$-10=(4)^2-4(4)-10$

$-10=16-16-10$

$-10=-10$, TRUE!

check! $(-3,11)$

$11=-(-3)^2-2(-3)+14$

$11=-9+6+14$

$11=-3+14$

$11=11$, TRUE!

$11=(-3)^2-4(-3)-10$

$11=9+12-10$

$11=21-10$

$11=11$, TRUE!

The solution set is

$\{(4,-10),(-3,11)\}$.

Solve using Elimination!

Ex 7. Solve the following system of equations: $\begin{cases} x^2+y^2=4 \\ y^2-x=4 \end{cases}$ $\rightarrow -1\cdot(y^2-x)=-1\cdot 4$

$-y^2+x=-4$

The solution set is $\{(0,2),(0,-2),(-1,\sqrt{3}),(-1,-\sqrt{3})\}$.

$$\begin{array}{r} x^2+y^2=4 \\ +\quad -y^2+x=-4 \\ \hline x^2+x=0 \end{array}$$

$x(x+1)=0$

Either

$x=0$, or $x+1=0$

$x=-1$

Find y:

$y^2-x=4$

$x=0$

$y^2-(0)=4$

$y^2=4$

$y=\sqrt{4}$, or $y=-\sqrt{4}$

$y=2$, or $y=-2$

$(0,2)$ & $(0,-2)$

$y^2-x=4$

$x=-1$

$y^2-(-1)=4$

$y^2+1=4$

$y^2+1-1=4-1$

$y^2=3$

$y=\sqrt{3}$, or $y=-\sqrt{3}$

$(-1,\sqrt{3})$ & $(-1,-\sqrt{3})$

check: $(0,2)$

$(0)^2+(2)^2=4$

$4=4$, TRUE!

$(2)^2-(0)=4$

$4=4$, TRUE!

check: $(0,-2)$

$(0)^2+(-2)^2=4$

$4=4$, TRUE!

$(-2)^2-(0)=4$

$4=4$, TRUE!

check: $(-1,\sqrt{3})$

$(-1)^2+(\sqrt{3})^2=4$

$1+3=4$

$4=4$, TRUE!

check: $(-1,-\sqrt{3})$

$(-1)^2+(-\sqrt{3})^2=4$

$1+3=4$

$4=4$, TRUE!

Ex 8. Solve the following system of equations: $\begin{cases} x^2+y=4 \\ 2x+y=1 \end{cases}$ $\rightarrow -1\cdot(2x+y)=-1\cdot 1$

solve using Elimination!

$-2x-y=-1$

$$\begin{array}{r} +\quad x^2+y=4 \\ -2x-y=-1 \\ \hline x^2-2x=3 \end{array}$$

$x^2-2x-3=3-3$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

Either

$x-3=0$, or $x+1=0$

$x=3$, or $x=-1$

check: $(3,-5)$

$(3)^2+(-5)=4$

$9-5=4$

$4=4$, TRUE!

$2(3)+(-5)=1$

$6-5=1$

$1=1$, TRUE!

check: $(-1,3)$

$(-1)^2+(3)=4$

$1+3=4$

$4=4$, TRUE!

$2(-1)+(3)=1$

$-2+3=1$

$1=1$, TRUE!

Find y:

$2x+y=1$

$x=3$

$2(3)+y=1$

$6+y=1$

$-6+6+y=-6+1$

$y=-5$

$(3,-5)$

$2x+y=1$

$x=-1$

$2(-1)+y=1$

$-2+y=1$

$-2+y+2=1+2$

$y=3$

$(-1,3)$

The solution ~~set~~ is $\{(3,-5),(-1,3)\}$.

Solve using Elimination:

Ex 9. Solve the following system of equations: $\begin{cases} x^2+(y-2)^2=4 \\ x^2-2y=0 \end{cases}$ $\longrightarrow$ $-1\cdot(x^2-2y)=-1\cdot 0$

$-x^2+2y=0$

$$\begin{aligned} & x^2+(y-2)^2=4 \\ + \; & -x^2+2y=0 \end{aligned}$$

$(y-2)^2+2y=4$

$(y-2)(y-2)+2y=4$

$y^2-4y+4+2y=4$

$y^2-2y+4=4$

$y^2-2y+4-4=4-4$

$y^2-2y=0$

$y\cdot(y-2)=0$

Either

$y=0$, or $y-2=0$

$y=2$

Find x!

$x^2-2y=0$

$y=0$

$x^2-2(0)=0$

$x^2=0$

$x=0$

$(0,0)$

$x^2-2y=0$

$y=2$

$x^2-2(2)=0$

$x^2-4=0$

$x^2-4+4=0+4$

$x^2=4$

Either

$x=\sqrt{4}$, or $x=-\sqrt{4}$

$x=2$, or $x=-2$

$(2,2)$ & $(-2,2)$

check: $(0,0)$

$(0)^2+[(0)-2]^2=4$

$0+[-2]^2=4$

$4=4$, TRUE!

$(0)^2-2(0)=0$

$0-0=0$

$0=0$, TRUE!

check: $(2,2)$

$(2)^2+[(2)-2]^2=4$

$4+(0)^2=4$

$4=4$, TRUE!

$(2)^2-2(2)=0$

$4-4=0$

$0=0$, TRUE!

check: $(-2,2)$

$(-2)^2+[(2)-2]^2=4$

$4+(0)^2=4$

$4=4$, TRUE!

$(-2)^2-2(2)=0$

$4-4=0$

$0=0$, TRUE!

The solution set is $\{(0,0), (2,2), (-2,2)\}$.